

## The Modular Group of the form : $M_{2^n} \times C_2$

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**Abstract.** In this paper, the classification of finite  $p$ -groups is extended to the group of the modular structure  $M_{2^n} \times C_2$ , and the number of distinct subgroups were computed, making the classification of the given structure possible for the given prime  $p = 2$

**Keywords:** Finite  $p$ -groups, nilpotent group, fuzzy subgroups, Dihedral group, inclusion-exclusion principle, maximal subgroups

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### 1. Introduction

The classifications in finite  $p$ -groups is fast gaining considerable grounds. Of recent, various contributions have been made of which many viable, resourceful, beneficia and enriching l publications have been made. In [1], the case of dihedral was considered while [2] dealt with the generalised quaternion. In [3], the work was extended to the quasidihedral (otherwise referred to as the semidihedral groups), and In [4], efforts are being made to close these aspect up by generalising the concepts up to the entire prime  $p$  i.e including both the even as well as the odd cases. Many works on BCI/BCK algebras and related algebras are available in literature, some of them are discussed in [7-15].

The following properties for the fuzzy subgroups of  $G$  were known.

1. The level sets of a fuzzy subset of a finite set form a chain .
2.  $\lambda$  is a fuzzy subgroup of  $G$  iff its level sets are subgroups of  $G$ '
3. The relation  $\sim$  is an equivalence relation on fuzzy subgroups of  $G$ , where for fuzzy subgroups  $\mu, \nu$  of  $G$ ,  $\mu \sim \nu$  iff  $\forall x, y \in G, (\mu(x) > \mu(y) \text{ iff } \nu(x) > \nu(y))$ .

### 2. Preliminaries

Suppose that  $(G, \cdot, e)$  is a group with identity  $e$ . Let  $S(G)$  denote the collection of all fuzzy subsets of  $G$ . An element  $\lambda \in S(G)$  is said to be a fuzzy subgroup of  $G$  if the following two conditions are sat.

1.  $\lambda(ab) \geq \min\{\lambda(a), \lambda(b)\}, \forall a, b \in G;$
2.  $\lambda(a^{-1}) \geq \lambda(a)$  for any  $a \in G$ .

And, since  $(a^{-1})^{-1} = a$ , we have that  $\lambda(a^{-1}) = \lambda(a)$ , for any  $a \in G$ .

Also, by this notation and definition,  $\lambda(e) = \sup\lambda(G)$ . [Marius [5]].

Now, concerning the subgroups, the set  $FL(G)$  possessing all fuzzy subgroups of

$G$  forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of  $G$ .

In what follows, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite  $p$ -group  $G$  is described. ( See [2] and [3] ) Suppose that  $M_1, M_2, \dots, M_t$  are the maximal subgroups of  $G$ . Let  $h(G)$  denote the number of chains of subgroups of  $G$  which ends in  $G$ . The method of computing  $h(G)$  is based on the application of the Inclusion-Exclusion Principle. If  $A$  is the set of chains in  $G$  of type  $C_1 \subset C_2 \subset \dots \subset C_r = G$ , and  $A'$  represents the set of chains of  $A'$  which are contained in  $M_r$ ,  $r = 1, \dots, t$ .

Then we have:

$$\begin{aligned} |A| &= 1 + |A'| = |\cup_{r=1}^t A_r| \\ &= 1 + \sum_{r=1}^t |A_r| - \sum_{1 \leq r_1 < r_2 \leq t} |A_{r_1} \cap A_{r_2}| + \dots + (-1)^{t-1} |\cap_{r=1}^t A_r| \end{aligned}$$

Observe that, for every  $1 \leq w \leq t$  and  $1 \leq r_1 < r_2 < \dots < r_w \leq t$ , the set  $\cap_{i=1}^w A_{r_i}$  consists of all chains of  $A'$  which are included in  $\cap_{i=1}^w M_{r_i}$ . We have that

$$|\cap_{i=1}^w A_{r_i}| = 2h(\cap_{i=1}^w M_{r_i})^{-1}$$

$$\therefore |A| = 1 + \sum_{r=1}^t (2h(M_r) - 1) - \sum_{1 \leq r_1 < r_2 \leq t} (2h(M_{r_1} \cap M_{r_2}) - 1)$$

$$+ \dots + (-1)^{t-1} (2h(\cap_{r=1}^t M_r) - 1)$$

$$= 2 \left( \sum_{r=1}^t h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \dots + (-1)^{t-1} h(\cap_{r=1}^t M_r) \right) + C$$

And

$$\begin{aligned} C &= 1 + \sum_{r=1}^t (-1) - \sum_{1 \leq r_1 < r_2 \leq t} (-1) + \dots + (-1)^{t-1} (-1) \\ &= (1 - 1)^t = 0 \end{aligned}$$

we have that:

$$\begin{aligned} h(G) &= 2 \left( \sum_{r=1}^t h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) \right. \\ &\quad \left. + \dots + (-1)^{t-1} h(\cap_{r=1}^t M_r) \right) (3) \end{aligned}$$

In [6], (3) was used to obtain the explicit formulas of  $h(D_{2n})$  for some positive integers  $n$ . (Also see [1]) and [2]

**Theorem 2. [5]** The number of distinct fuzzy subgroups of a finite  $p$ -group of order  $p^n$  which have a cyclic maximal subgroup is:

1.  $h(\mathbb{Z}_{p^n}) = 2^n$
2.  $h(D_{2^n}) = 2^{2n-1}$
3.  $h(\varphi_{2^n}) = 2^{2n-2}$
4.  $h(S_{2^n}) = 3 \cdot 2^{2n-3}$
5.  $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1} [2 + (n-1)p]$

### 3. The nilpotent 2-group of the form $M_{2^n} \times C_2$

Recall that

$$M_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{1+2^{n-2}} \rangle$$

The Modular Group of the form :  $M_{2^n} \times C_2$

Since  $n = 3$  is not defined for this particular structure, we begin by taking the case,  $n = 4$ . We have:

$$M_{2^4} = \langle x, y | x^8 = y^2 = 1, y^{-1}xy = x^5 \rangle$$

$$= \left\{ \begin{array}{l} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, \\ x^2y, x^3y, x^4y, x^5y, x^6y, x^7y \end{array} \right\}$$

$$\text{From here, } M_{2^4} \times C_2 = \left\{ \begin{array}{l} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, \\ x^2y, x^3y, x^4y, x^5y, x^6y, x^7y \end{array} \right\} \times \{1, a\}$$

$$= \left\{ \begin{array}{l} (1,1), (1, a), (x, 1), (x, a), (x^2, 1), (x^2, a), (x^3, 1), (x^3, a), (x^4, 1), (x^4, a), \\ (x^5, 1), (x^5, a), (x^6, 1), (x^6, a), (x^7, 1), (x^7, a), (y, 1), (y, a), (xy, 1), (xy, a), \\ (x^2y, 1), (x^2y, a), (x^3y, 1), (x^3y, a), (x^4y, 1), (x^4y, a), (x^5y, 1), (x^5y, a), \\ (x^6y, 1), (x^6y, a), (x^7y, 1), (x^7y, a) \end{array} \right\}$$

We have the maximal subgroups for  $M_{2^4} \times C_2$  as follows:

$$M_1 = \left\{ \begin{array}{l} (1,1), (1, a), (x, 1), (x, a), (x^2, 1), (x^2, a), (x^3, 1), (x^3, a), (x^4, 1), (x^4, a), \\ (x^5, 1), (x^5, a), (x^6, 1), (x^6, a), (x^7, 1), (x^7, a), \end{array} \right\}$$

$$M_2 = \left\{ \begin{array}{l} (1,1), (x, 1), (x^2, 1), (x^3, 1), (x^4, 1), (x^5, 1), (x^6, 1), (x^7, 1), \\ (y, 1), (xy, 1), (x^2y, 1), (x^3y, 1), (x^4y, 1), (x^5y, 1), (x^6y, 1), (x^7y, 1) \end{array} \right\}$$

$$M_3 = \left\{ \begin{array}{l} (1,1), (x, 1), (x^2, 1), (x^3, 1), (x^4, 1), (x^5, 1), (x^6, 1), (x^7, 1), \\ (y, a), (xy, a), (x^2y, a), (x^3y, a), (x^4y, a), (x^5y, a), (x^6y, a), (x^7y, a) \end{array} \right\}$$

$$M_4 = \left\{ \begin{array}{l} (1,1), (x, a), (x^2, 1), (x^3, a), (x^4, 1), (x^5, a), (x^6, 1), (x^7, a), \\ (y, a), (xy, 1), (x^2y, a), (x^3y, 1), (x^4y, a), (x^5y, 1), (x^6y, a), (x^7y, 1) \end{array} \right\}$$

$$M_5 = \left\{ \begin{array}{l} (1,1), (x, a), (x^2, 1), (x^3, a), (x^4, 1), (x^5, a), (x^6, 1), (x^7, a), \\ (y, 1), (xy, a), (x^2y, 1), (x^3y, a), (x^4y, 1), (x^5y, a), (x^6y, 1), (x^7y, a) \end{array} \right\}$$

$$M_6 =$$

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$$\left\{ \begin{array}{l} (1,1), (1, a), (x^2, 1), (x^2, a), (x^4, 1), (x^4, a), (x^6, 1), (x^6, a), \\ (y, 1), (y, a), (x^2y, 1), (x^2y, a), (x^4y, 1), (x^4y, a), (x^6y, 1), (x^6y, a) \end{array} \right\}$$

$$M_7 = \left\{ \begin{array}{l} (1,1), (1, a), (x^2, 1), (x^2, a), (x^4, 1), (x^4, a), (x^6, 1), (x^6, a), \\ (xy, 1), (xy, a), (x^3y, 1), (x^3y, a), (x^5y, 1), (x^5y, a), (x^7y, 1), (x^7y, a) \end{array} \right\}$$

From here,  $M_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2^3$ ,

$$M_2 \cong M_3 \cong M_4 \cong M_5 \cong M_7 \cong M_{2^4}$$

And,  $M_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^2$ .

By making the application of equation (c), we have,

$$\begin{aligned} \frac{1}{2}h(M_{2^4} \times C_2) &= [2h(\mathbb{Z}_2 \times \mathbb{Z}_2^3) + 4h(M_{2^4}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^2)] \\ &\quad - 3[h(\mathbb{Z}_2^3) + h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) + 2h(\mathbb{Z}_2^3) + 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^2)] \\ &\quad + [28h(\mathbb{Z}_2^2) + 2h(\mathbb{Z}_2^3) + h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) + 2h(\mathbb{Z}_2^3) \\ &\quad + 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^2)] - 35h(\mathbb{Z}_2^2) + 21h(\mathbb{Z}_2^2) - 7h(\mathbb{Z}_2^2) + h(\mathbb{Z}_2^2) \\ &= 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^3) + 4h(M_{2^4}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^2) \\ &\quad - 8h(\mathbb{Z}_2^3) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) + 8h(\mathbb{Z}_2^2) \\ &= 4(64) + 2(64) + 304 - 8(8) - 6(24) + 8(4) \\ &= 320 + 304 + 32 - 144 \\ &= 512. \end{aligned}$$

$$\therefore h(M_{2^4} \times C_2) = 2 \times 512 = 1024.$$

**The number of fuzzy subgroups of the nilpotent group :  $M_{2^5} \times C_2$**

$$\begin{aligned} \frac{1}{2}h(M_{2^5} \times C_2) &= [2h(\mathbb{Z}_2 \times \mathbb{Z}_2^4) + 4h(M_{2^5}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^3)] \\ &\quad - 3[h(\mathbb{Z}_2^4) + h(\mathbb{Z}_2 \times \mathbb{Z}_2^3) + 2h(\mathbb{Z}_2^4) + 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^3)] \\ &\quad + [28h(\mathbb{Z}_2^3) + 2h(\mathbb{Z}_2^4) + h(\mathbb{Z}_2 \times \mathbb{Z}_2^3) + 2h(\mathbb{Z}_2^4) + 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^3)] \\ &\quad - 35h(\mathbb{Z}_2^3) + 21h(\mathbb{Z}_2^3) - 7h(\mathbb{Z}_2^3) + h(\mathbb{Z}_2^3) \\ &= 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^4) + 4h(M_{2^5}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^3) \\ &\quad - 8h(\mathbb{Z}_2^4) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_2^3) + 8h(\mathbb{Z}_2^3) \\ &= h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^3) + 2(160) + 4(160) - 8(16) - 6(64) \times 8(8) \\ &= h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^3) + 512 \\ &= 1024 + 512 = 1536 \\ \therefore h(M_{2^5} \times C_2) &= 2 \times 1536 \\ &= 3072. \end{aligned}$$

**4. The number of fuzzy subgroups of the nilpotent group:  $M_{2^n} \times C_2$**

In general, for any positive integer  $n > 4$

$$\begin{aligned} h(M_{2^n} \times C_2) &= 2 \left[ \begin{array}{l} 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^{n-1}) + 4h(M_{2^n}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{n-2}) \\ - 8h(\mathbb{Z}_2^{n-1}) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_2^{n-2}) + 8h(\mathbb{Z}_2^{n-2}) \end{array} \right] \\ &= 2h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{n-2}) + 2^{n+1}(3n + 1) \end{aligned}$$

$$\begin{aligned} & \text{The Modular Group of the form : } M_{2^n} \times C_2 \\ & = 2^n(3n^2 - n - 6) + 2^{n+1}(3n + 1) \end{aligned}$$

$$\begin{aligned} h(M_{2^n} \times C_2) &= 2^n(3n^2 + 5n - 4), \quad n > 4 \\ &= 2^n \left[ 3 \left( n + \frac{5}{6} \right)^2 - \frac{73}{12} \right], \quad p = 2 \end{aligned}$$

## 5. Conclusion

Finally, the classification for the nilpotent 2-groups of the specified modular structure given by :  $M_{2^n} \times C_2$  is thus hereby clearly made with the number of distinct fuzzy subgroups explicitly computed for the prime  $p = 2$  and every non zero integer  $n \geq 3$

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