

The Subgroups for the Finite p -Group of the Structure $D_{2^4} \times C_{2^5}$

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Received 20 October 2022; accepted 20 December 2022

ABSTRACT

Any finite nilpotent group can be uniquely written as a direct product of p -groups. In this paper, an attempt for the computation of $D_{2^4} \times C_{2^5}$ was made. This happens as the computation of the number of distinct fuzzy subgroups of Cartesian product of the dihedral group of order 2^4 with another cyclic group having an order 2^5 .

Keywords: Finite p -groups, nilpotent group, fuzzy subgroups, dihedral group, inclusion-exclusion principle, maximal subgroups

AMS Mathematics Subject Classification (2010): 08A72, 20D15, 60A86

1. Introduction

Dynamism is always the order of the day, most generally in the aspect of pure Mathematics over the years. For instance, many researchers have treated cases of finite abelian groups. Since then, the study has been extended to other important classes of finite abelian and nonabelian groups, such as the dihedral, quaternion, semi dihedral, and hamiltonian groups. This work actually calculate the exact number of distinct fuzzy subgroups for nilpotent groups derived from p -groups. The use of the exclusion – inclusion method have been involved in a number of related research directions, such as we had in [1]. Our esteemed reader can as well consult [3 to 17] of our references.

2. Methodology

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p -group G is described. Suppose that M_1, M_2, \dots, M_t are the maximal subgroups of G , and denote by $h(G)$ the number of chains of subgroups of G which ends in G . By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2 \left(\sum_{r=1}^t h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \dots + (-1)^{t-1} h \left(\bigcap_{r=1}^t M_r \right) \right) (\#)$$

In [2], (#) was used to obtain the explicit formulas for some positive integers n .

Theorem 2.1. (*) [Marius]: The number of distinct fuzzy subgroups of a finite p -group of order p^n which have a cyclic maximal subgroup is:

1. $h(\mathbb{Z}_{p^n}) = 2^n$
2. $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

3. The number of fuzzy subgroups for $\mathbb{Z}_8 \times \mathbb{Z}_8$

Lemma 3.1. (γ): Let G be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$.

Then, $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$

Proof: By the use of GAP (Group Algorithms and Programming), G has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = h(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

And by theorem (*), $h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 24 \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48$

Corrolary 3.2. Following the last lemma, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5})$, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6})$, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ and $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8}) = 1536, 4096, 10496$ and 26112 respectively.

Theorem 3.3. Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$, then $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$

Proof: The three maximal subgroups of G have the following properties :

one is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}$, while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n}$.

We have : $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) = h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$

Hence, $h(G) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) + 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}}) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) + 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) + \dots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}}) + 2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}})$, for $n-j = 3$

$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3} [2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})]$
 $= 2^{n+2}[n^2 + 5n + 3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) = 2^{n+2}((n^2 + 5n + 3) + \frac{1}{6}(n-3)(n^2 + 9n + 14)) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24), n > 2.$

Theorem 3.4. Suppose that $G = D_{2^3} \times \mathbb{C}_8$. Then, $h(G) = 5376$

Proof: $\frac{1}{2}h(G) = h(D_{2^3} \times \mathbb{Z}_4) + 2h(\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2) - 4h(\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2) + h(\mathbb{Z}_8 \times \mathbb{Z}_4) - 6h(\mathbb{Z}_8 \times \mathbb{Z}_2) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_4) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_2) + h(\mathbb{Z}_{2^3}) = 2688 \therefore h(G) = 2.2688 = 5376.$

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Theorem 3.5. Let $G = D_{2^5} \times \mathbb{Z}_8$. Then, $h(G) = 111136$

Proof: $\frac{1}{2}h(G) = h(D_{2^5} \times \mathbb{Z}_{2^2}) + 2h(D_{2^4} \times \mathbb{Z}_{2^3}) - 4h(D_{2^4} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^4}) - 4h(\mathbb{Z}_{2^3}) = 55568.$
 $\therefore h(G) = 2 \times 55568 = 111136.$

Theorem 3.6. Suppose that $G = D_{2^6} \times \mathbb{Z}_8$. Then, $h(G) = 492864$

Proof: $\frac{1}{2}h(G) = h(D_{2^6} \times \mathbb{Z}_4) + 2h(D_{2^5} \times \mathbb{Z}_{2^3}) - 4h(D_{2^5} \times \mathbb{Z}_4) + h(\mathbb{Z}_{2^5} \times \mathbb{Z}_{2^3}) - 2h(\mathbb{Z}_{2^5} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^5}) - 4h(\mathbb{Z}_{2^4}) = 246432.$
 $\therefore h(G) = 2 \times 246432 = 492864.$

Theorem 3.7. Let $G = D_{2^n} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by : $h(G) = 2^{2n}(2n + 1) - 2^{n+1}, n > 3$

4. The number of fuzzy subgroups for $D_{2^n} \times \mathbb{C}_8$

Proposition 4.1. Suppose that $G = D_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of G is given by :

$$2^{2(n-1)}(6n + 113) + 2^n[13 - 6n - 2n^2 + 3 \sum_{j=1}^{n-3} 2^{(j-1)}(2n + 1 - 2j)]$$

$$+ \frac{1}{3}(2^{n+2})[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 + \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]]$$

Proof: $h(D_{2^n} \times \mathbb{C}_8) = 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times \mathbb{Z}_4) + 2h(D_{2^{n-1}} \times \mathbb{C}_8) + 4h(\mathbb{Z}_{2^{n-2}} \times \mathbb{C}_8) + 2^4h(\mathbb{Z}_{2^{n-3}} \times \mathbb{C}_8) + 2^6h(\mathbb{Z}_{2^{n-4}} \times \mathbb{C}_8) - 2^8h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) - 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10}h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^2}) - 2^9h(\mathbb{Z}_{2^{n-5}}) - 2^9h(D_{2^{n-4}} \times \mathbb{C}_{2^2}) + 2^8h(D_{2^{n-4}} \times \mathbb{C}_{2^3}) = 2^n + 2h(D_{2^n} \times \mathbb{C}_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) - 2^{2(n-3)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}) - 2^2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5}h(\mathbb{Z}_{2^2}) - 2^{2n-5}h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)}h(D_{2^3} \times \mathbb{Z}_{2^3}) + 3 \sum_{i=1}^{n-5} 2^{2ij}h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3})$
as required. ■

Proposition 4.2. (see [16]) Suppose that $G = D_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of G is given by :

$$2^{2(n-1)}(6n + 113) + 2^n[13 - 6n - 2n^2 + 3 \sum_{j=1}^{n-3} 2^{(j-1)}(2n + 1 - 2j)]$$

$$+ \frac{1}{3}(2^{n+2})[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 + \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]]$$

Proof: $h(D_{2^n} \times \mathbb{C}_8) = 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times \mathbb{Z}_4) + 2h(D_{2^{n-1}} \times \mathbb{C}_8) + 4h(\mathbb{Z}_{2^{n-2}} \times \mathbb{C}_8) + 2^4h(\mathbb{Z}_{2^{n-3}} \times \mathbb{C}_8) + 2^6h(\mathbb{Z}_{2^{n-4}} \times \mathbb{C}_8) - 2^8h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) - 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) +$

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$$\begin{aligned}
 & 2^{10}h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_2) - 2^9h(\mathbb{Z}_{2^{n-5}}) - 2^9h(D_{2^{n-4}} \times C_{2^2}) + 2^8h(D_{2^{n-4}} \times C_{2^3}) \\
 &= 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2) + 2^2h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_2) - 2^{2(n-3)}h(\mathbb{Z}_2 \times \mathbb{Z}_2) \\
 &\quad + 2^{2(n-2)}h(\mathbb{Z}_2 \times \mathbb{Z}_2) - 2^2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2) - 2^{2n-5}h(\mathbb{Z}_2) \\
 &\quad - 2^{2n-5}h(D_{2^3} \times \mathbb{Z}_2) + 2^{2(n-3)}h(D_{2^3} \times \mathbb{Z}_2) \\
 &\quad + \sum_{i=1}^{n-5} 3 \cdot 2^{2i}h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_2)
 \end{aligned}$$

as required.

Theorem 4.3. (see [3]) Let $G = D_{2^4} \times C_{2^4}$. Then, $h(G) = 61384$.

Proof: There exist seven maximal subgroups. Two isomorphic to $D_{2^4} \times C_{2^3}$, two isomorphic to $D_{2^3} \times C_{2^4}$, two isomorphic to $D_{2^4} \times C_{2^2}$, while the seventh is isomorphic to \mathbb{Z}_{2^4} .

$$\begin{aligned}
 & \text{Hence, we have that: } \frac{1}{2}h(G) = 2h(D_{2^4} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) + 2h(D_{2^3} \times \\
 & Z_{2^4}) - 6h(D_{2^3} \times \mathbb{Z}_2) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_2) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_2) - 6h(\mathbb{Z}_2) + 2h(D_{2^3} \times \\
 & \mathbb{Z}_2) + 28h(\mathbb{Z}_2 \times \mathbb{Z}_2) + 2h(Z_{2^4} \times Z_{2^2}) + 2h(\mathbb{Z}_2) + h(Z_{2^3} \times \mathbb{Z}_2) - 35h(\mathbb{Z}_2 \times \\
 & \mathbb{Z}_2) + 21h(\mathbb{Z}_2 \times \mathbb{Z}_2) - 7h(\mathbb{Z}_2 \times \mathbb{Z}_2) + h(\mathbb{Z}_2 \times \mathbb{Z}_2) \\
 &= 2[h(D_{2^4} \times Z_{2^2}) + h(D_{2^4} \times Z_{2^3}) + h(D_{2^3} \times Z_{2^4}) - 2h(D_{2^3} \times \mathbb{Z}_2) - \\
 & 2h(\mathbb{Z}_2 \times \mathbb{Z}_2) - h(\mathbb{Z}_2 \times \mathbb{Z}_2) + 4h(D_{2^3} \times \mathbb{Z}_2) - 3h(\mathbb{Z}_2) + \frac{1}{2}h(Z_{2^4})] \\
 &\therefore h(G) = 4[700 + 8416 + 10744 - 10752 - 1088 + 162 + 704 - 40] = 4[15346] \\
 &= 61384 \quad \blacksquare
 \end{aligned}$$

5. The number of fuzzy subgroups for $D_{2^5} \times C_{2^4}$

Proposition 5.1. Let $G = (D_{2^4} \times C_{2^5})$. Then, we are going to have that

$$\begin{aligned}
 & \frac{1}{2}h(G) = h(D_{2^4} \times C_{2^4}) + 3h(C_{2^3} \times C_{2^5}) + 2h(C_{2^5} \times C_{2^3}) - 4h(D_{2^4} \times C_{2^3}) - 6h(C_{2^4} \\
 & \times C_{2^3}) + 8h(C_{2^3} \times C_{2^3}) - 3h(C_{2^4}) \\
 &= 3h(D_{2^5} \times C_{2^3}) + 3h(D_{2^4} \times C_{2^4}) - 4h(D_{2^4} \times C_{2^3}) - 6h(C_{2^3} \times C_{2^4}) + \\
 & 8h(C_{2^3} \times C_{2^3}) - 3h(C_{2^4}) \\
 & \text{Now, } h(G) = 6h(D_{2^5} \times C_{2^3}) + 6h(D_{2^4} \times C_{2^4}) - 8h(D_{2^4} \times C_{2^3}) - 12h(C_{2^3} \times \\
 & C_{2^4}) + 16h(C_{2^3} \times C_{2^3}) - 6h(C_{2^4}).
 \end{aligned}$$

Acknowledgements. The authors are grateful to the reviewers for their valuable comments for improvement of the paper.

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