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Research article

The exponentiated generalized power series Family of distributions: theory, properties and applications

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ABSTRACT

We propose a new generalized family of distributions called the exponentiated generalized power series (EGPS) family of distributions and study its sub-model, the exponentiated generalized logarithmic (EGL) class of distributions, in detail. The structural properties of the new model (EGPS) and its sub-model (EGL) distribution including moments, order statistics, Rényi entropy, and maximum likelihood estimates are derived. We used the method of maximum likelihood to estimate the parameters of this new family of distributions. Simulation study was carried out to examine the bias and the mean square error of the maximum likelihood estimators for each of the model's parameters. Finally, we showed real life data examples to illustrate the models' applicability, flexibility and usefulness.

1. Introduction

Several new classes of distributions can be developed by adding one or more parameters to an existing distribution or family of distributions that have been proposed in the statistical literature including the Marshall-Olkin generated by Marshall and Olkin (1997), beta-G by Eugene, Lee and Famoye (2002), Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011), McDonald-G (Mc-G) by Alexander et al. (2012), transformer (T-X) by Alzaghal, Famoye and Lee (2013), Weibull-G by Bourguignon, Silva and Cordeiro (2014), exponentiated half-logistic by Cordeiro, Ortega and da Cunha (2013), Logistic-X by Tahir et al. (2016), Lomax generator by Cordeiro et al. (2014), Kumaraswamy odd log-logistic family by Alizadeh et al. (2015a), Kumaraswamy Marshall-Olkin family by Alizadeh et al. (2015b) and Type-1 half-logistic family of distributions by Cordeiro, Alizadeh and Diniz (2015). Determining an adequate model for inferential purposes is a very important problem in statistical modeling. Barreto-Souza, Lemonto and Cordeiro (2013) stated that addition of parameters to a well-established distribution is a proven technique to obtain more flexible and diverse new families of distributions.

Cordeiro, Ortega and da Cunha (2013) introduced a class of distributions called the exponentiated generalized class of distributions (EG) which allows for greater flexibility of its tails and is widely applied in many areas of engineering and biology. The EG class extends several common distributions studied recently such as the exponentiated exponential, exponentiated Weibull, exponentiated gamma, exponentiated Fréchet and exponentiated Gumbel distributions. An attractive feature about the model is that the two extra parameters that are introduced have the capabilities to control both the weights at the tails and possibly add entropy to the center of the EG density function. There are various univariate distributions and recent developments focus on constructing flexible distributions from classic ones to facilitate better modeling of lifetime data and other types of data.

In this paper, we propose a new class of distributions, referred to as the exponentiated generalized power series class of distributions (EGPS), and its specific case the exponentiated generalized logarithmic class of distributions (EGL), which contains special several well known distributions as some of its sub-models. The exponentiated generalized logarithmic (EGL) class of distributions, is obtained by mixing exponentiated generalized class of distributions and logarithmic distribution. The generalized distributions can accommodate many forms of the hazard function and contain several well-known distributions as some special sub-models. An attractive feature about the generalized distributions is that they can model different

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types of failure rate functions that is increasing, decreasing, unimodal and bath-tub depending on its parameters. These cases are very common in reliability, engineering and biological studies.

We present this paper as follows. In section 2, the generalized distribution and its probability density function (pdf)is presented. Some structural properties of the generalized model consisting of the hazard, reverse hazard and quantile functions, moments, conditional moments and moment generating function are derived and presented. Furthermore, the distribution of the order statistics, L-moments and Rényi entropy and estimates of model parameters are derived. In section 3, we introduce the exponentiated generalized logarithmic (EGL) class including the structural properties such as the hazard, reverse hazard, and quantile functions, various sub-models, moments, conditional moments and moment generating function, the order statistics, L-moments and Rényi entropy. The estimates of the parameters of the new model and the comparisons of these estimates with estimates from other models are presented in section 3. The Monte Carlo (MCMC) simulation study was conducted to examine the bias and mean square error of the maximum likelihood estimators for each parameter in section 4. We applied proposed model to real data in section 5, followed by conclusions.

2. The model, sub-models and properties

In this section, we developed the new distribution called Exponentiated Generalized Power Series (EGPS) distribution and derived some of its properties.

Cordeiro, Ortega and da Cunha (2013) studied and developed the exponentiated generalized (EG) class of distributions with cdf and pdf given by:

$$F_{EG}(x) = \left[1-(1-G(x;\xi))^{\alpha}\right]^{\beta} \tag{2.1}$$

and

$$f_{EG}(x) = \alpha\beta\left[(1-G(x;\xi))\right]^{\alpha-1}\left[1-(1-G(x;\xi))^{\alpha}\right]^{\beta-1} g(x;\xi), \tag{2.2}$$

respectively, where $x > 0$, $\alpha > 0$, and $\beta > 0$.

Let N, be a zero truncated discrete random variable (RV) with a power series distribution, whose probability mass function (pmf) is given by

$$P(N=n) = \frac{a_n\theta^n}{C(\theta)}, n = 1,2,3,..., \tag{2.3}$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions. Given N, let $X_1, X_2, ..., X_N$ be identically and independently distributed (iid) random variable following EG class of distributions. Let $X_{(n)} = \max(X_1, X_2, ..., X_N)$. Then the cumulative distribution function (cdf) of $X_{(n)}|N=n$ is given by

$$F_{X_{(n)}|N=n}(x;\alpha,\beta,\xi) = \left(1-\left(1-G(x;\xi)\right)^{\alpha}\right)^{n\beta}, \alpha,\beta,\xi > 0, n \geq 1. \tag{2.4}$$

The Exponentiated Generalized Power Series distribution denoted by EGPS is defined by the marginal distribution of $X_{(n)}$ and is given by

$$\begin{aligned} F_{X_{(n)}}(x) &= \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)}\left(\left(1-(1-G(x;\xi))^{\alpha}\right)\right)^{n\beta} \\ &= \frac{C\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)}{C(\theta)}, \quad x > 0. \end{aligned} \tag{2.5}$$

The corresponding pdf is given by

$$\begin{aligned} f_{X_{(n)}}(x) &= \theta\alpha\beta g(x;\xi)\left(1-\left(1-G(x;\xi)^{\alpha}\right)\right)^{\beta-1}\left(1-G(x;\xi)\right)^{\alpha-1} \\ &\times \frac{C'\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)}{C(\theta)}. \end{aligned} \tag{2.6}$$

The hazard and reverse hazard functions are given by

$$h_F(x) = \frac{\theta\alpha\beta g(x;\xi)\left(1-\left(1-G(x;\xi)\right)^{\alpha}\right)^{\beta-1}\left(1-G(x;\xi)\right)^{\alpha-1} C'\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)}{C(\theta)-C\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)} \tag{2.7}$$

and

$$\tau_F(x) = \frac{\theta\alpha\beta g(x;\xi)\left(1-\left(1-G(x;\xi)\right)^{\alpha}\right)^{\beta-1}\left(1-G(x;\xi)\right)^{\alpha-1} C'\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)}{C\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)}, \tag{2.8}$$

respectively.

2.1. Quantile function

Let X be a random variable with cdf as in equation (2.5). The quantile function $Q_{X_{(n)}}(u)$ is defined by $G_{X_{(n)}}(Q_{X_{(n)}}(u)) = u, 0 \leq u \leq 1$. Therefore,

$$Q_{X_{(n)}}(u) = G^{-1}\left[1-\left(1-\left(\frac{C^{-1}(uC(\theta))}{\theta}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]. \tag{2.9}$$

2.2. Expansion of density

Expansion of the density of the EGPS distribution is presented in this sub-section. We can rewrite the EGPS density function as follows:

$$\begin{aligned}
f_{X_{(n)}}(x) &= \theta\alpha\beta g(x;\xi)\Big(1-\big(1-G(x;\xi)\big)^{\alpha}\Big)^{\beta-1}\big(1-G(x;\xi)\big)^{\alpha-1} \\
&\quad\times \frac{C'\Big(\theta\big(1-(1-G(x;\xi))^{\alpha}\big)^{\beta}\Big)}{C(\theta)} \\
&= \theta\alpha\beta g(x;\xi)\Big(1-\big(1-G(x;\xi)\big)^{\alpha}\Big)^{\beta-1}\big(1-G(x;\xi)\big)^{\alpha-1} \\
&\quad\times \frac{\sum_{n=1}^{\infty} n a_n \theta\Big(1-\big(1-G(x;\xi)\big)^{\alpha}\Big)^{\beta(n-1)}}{C(\theta)} \\
&= \frac{\sum_{n=1}^{\infty} a_n\theta^n}{C(\theta)} n\alpha\beta g(x;\xi)\big(1-G(x;\xi)\big)^{\alpha-1}\Big(1-\big(1-G(x;\xi)\big)\Big)^{n\beta-1} \\
&= \frac{\sum_{n=1}^{\infty} a_n\theta^n}{C(\theta)} f_{EG}(x;n\beta,\alpha).
\end{aligned} \tag{2.10}$$

The various mathematical properties of the EGPS distribution can be readily obtained from equation (2.10) and the corresponding properties of the EG distribution.

2.3. Moments

The rth moment of the EGPS class of distributions is given by

$$\begin{aligned}
E(X^r) &= \int_0^{\infty} x^r\theta\alpha\beta g(x;\xi)\Big(1-\big(1-G(x;\xi)\big)^{\alpha}\Big)^{\beta-1}\big(1-G(x;\xi)\big)^{\alpha-1} \\
&\quad\times \frac{C'\Big(\theta\big(1-(1-G(x;\xi))^{\alpha}\big)^{\beta}\Big)}{C(\theta)}dx \\
&= \frac{\sum_{n=1}^{\infty} a_n\theta^n}{C(\theta)} n\theta\alpha\beta\int_0^{\infty} x^r g(x;\xi)\Big(1-\big(1-G(x;\xi)\big)^{\alpha}\Big)^{n\beta-1}\big(1-G(x;\xi)\big)^{\alpha-1}dx \\
&= \frac{\sum_{n=1}^{\infty} a_n\theta^n}{C(\theta)}\int_0^{\infty} x^r f_{EG}(x;n\beta,\alpha)dx \\
&= n\alpha\beta\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_n\theta^n(-1)^{j+k}\Gamma(n\beta)\Gamma(\alpha(j+1))}{j!k!\Gamma(n\beta-j)\Gamma(\alpha(j+1)-k)}\int_0^{\infty} x^r G(x;\xi)^k g(x;\xi)dx.
\end{aligned}$$

2.4. Conditional moments

The rth conditional moments for the EGPS class of distributions are given by

$$\begin{aligned}
E(X^r|X>t) &= \frac{1}{\bar{F}_{EGPS}(t)}\int_t^{\infty} x^r f_{EGPS}(x)dx \\
&= \frac{n\alpha\beta}{\bar{F}_{EGPS}(t)}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_n\theta^n(-1)^{j+k}\Gamma(n\beta)\Gamma(\alpha(j+1))}{j!k!\Gamma(n\beta-j)\Gamma(\alpha(j+1)-k)} \\
&\quad\times\int_t^{\infty} x^r G(x;\xi)^k g(x;\xi)dx.
\end{aligned}$$

2.5. Order statistics, L-moments and Rényi entropy

The concept of entropy is very crucial to the study of information theory. The entropy of a RV is a function of its probability distribution. It is a good measure of randomness or uncertainty. The distribution of the order statistic, L-moments and Rényi entropy of the new model is presented in this subsection.

2.5.1. Order statistics

Order statistics is very useful in probability and statistics. In this subsection, the distribution of the ith order statistic of the EGPS class of distributions and the corresponding moments are derived. The pdf of the ith order statistic from the EGPS pdf $f_{EGPS}(x)$ is given by

$$g_{i:n}(x)=\frac{n!f_{EGPS}(x)}{(i-1)!(n-i)!}[F_{EGPS}(x)]^{i-1}[1-F_{EGPS}(x)]^{n-i}$$
$$=\frac{n!f_{EGPS}(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F_{EGPS}(x)]^{j+i-1}$$

by using the binomial expansion $[1-F_{EGPS}(x)]^{n-i}=\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m[F_{EGPS}(x)]^m$, so that

$$g_{i:n}(x)=\frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\binom{n-i}{m}\frac{(-1)^m}{m+i}(m+i)[F_{EGPS}(x)]^{m+i-1}f_{EGPS}(x)$$
$$=\sum_{m=0}^{n-i}w_{i,m}f_{m+i}(x),$$

where $f_{m+i}(x)$ is the pdf of the EGPS distribution with parameters θ,α,β,ξ and $(m+i)$, $B(.,.)$ is the beta function and the weights $w_{i,m}$ are given by

$$w_{i,m}=\frac{1}{B(i,n-i+1)}\frac{(-1)^m}{m+i}\binom{n-i}{m}=(-1)^m\binom{m+i-1}{m}\binom{n}{m+i}.$$

The tth moment of the ith order statistics from the EGPS class of distributions is derivable using the result of Barakat and Abdelkader (2004):

$$E(X_{i:n}^t)=t\sum_{j=n-i+1}^{n}(-1)^{j-n+i-1}\binom{j-1}{n-i}\binom{n}{j}\int_0^\infty x^{t-1}[1-F_{EGPS}(x)]^j\,dx. \tag{2.11}$$

Note that

$$\int_0^\infty x^{t-1}[\overline{F}_{EGPS}(x)]^j\,dx=\int_0^\infty x^{t-1}\left(1-\frac{C\left(\theta\left(1-(1-G(x;\xi))^\alpha\right)^\beta\right)}{C(\theta)}\right)^j dx$$
$$=\sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\int_0^\infty x^{t-1}\left(\frac{C\left(\theta\left(1-(1-G(x;\xi))^\alpha\right)^\beta\right)}{C(\theta)}\right)^k dx$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(C(\theta))^k}\binom{j}{k}\int_0^\infty x^{t-1}\left(\sum_{s=0}^{\infty}a_{s+1}\theta^{s+1}\left(1-(1-G(x;\xi))^\alpha\right)^{\beta(s+1)}\right)^k dx.$$

We apply the result of a power series raised to a positive integer, with $a_s=a_{s+1}\theta^s$, that is,

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^k=\sum_{s=0}^{\infty}b_{s,k}y^s, \tag{2.12}$$

where $b_{s,k}=(sa_0)^{-1}\sum_{l=1}^{s}[k(l+1)-s]a_l b_{s-l,k}$, and $b_{0,k}=a_0^k$, (Gradshteyn and Ryzhik (2000)), to obtain

$$\int_0^\infty x^{t-1}[\overline{F}_{EGPS}(x)]^j\,dx=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^k}{C(\theta)^k}\binom{j}{k}\int_0^\infty x^{t-1}\left(1-(1-G(x;\xi))^\alpha\right)^{\beta k}$$
$$\times\left(\sum_{s=0}^{\infty}a_{s+1}\theta^s\left(1-(1-G(x;\xi))^\alpha\right)^{\beta s}\right)^k dx$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^k}{C(\theta)^k}\binom{j}{k}\int_0^\infty x^{t-1}\left(1-(1-G(x;\xi))^\alpha\right)^{\beta k}$$
$$\times\sum_{s=0}^{\infty}b_{s,k}\left(1-(1-G(x;\xi))^\alpha\right)^{\beta s}dx$$
$$=\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}b_{s,k}\frac{(-1)^k\theta^k}{C(\theta)^k}\binom{j}{k}\int_0^\infty x^{t-1}\left(\left(1-(1-G(x;\xi))^\alpha\right)\right)^{\beta(k+s)}dx$$
$$=\sum_{k=0}^{\infty}\sum_{s,w=0}^{\infty}b_{s,k}\frac{(-1)^{k+w}\theta^k}{C(\theta)^k}\binom{j}{k}\binom{\beta(k+s)}{w}\int_0^\infty x^{t-1}\left(1-G(x;\xi)\right)^{\alpha w}dx$$
$$=\sum_{k=0}^{\infty}\sum_{s,w,z=0}^{\infty}b_{s,k}\frac{(-1)^{k+w+z}\theta^k}{C(\theta)^k}\binom{j}{k}\binom{\beta(k+s)}{w}\binom{\alpha w}{z}\int_0^\infty x^{t-1}\left(G(x;\xi)\right)^z dx.$$

2.5.2. L-moments

According to Hoskings (1990), the L-moments are the expectations derived from linear combinations of order statistics. They can be estimated if the mean of the distribution exits, irrespective of whether some higher moments exist or not. The L-moments are given by

$$\lambda_{k+1}=\frac{1}{k+1}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}E(X_{k+1-j:k+1}),\quad k=0,1,2,\ldots\ldots \tag{2.13}$$

The L-moments of the EGPS distribution can be derived from the equation (2.11). The first four L-moments are given by $\lambda_1=E(X_{1:1})$, $\lambda_2=\frac{1}{2}E(X_{2:2}-X_{1:2})$, $\lambda_3=\frac{1}{3}E(X_{3:3}-2X_{2:3}+X_{1:3})$ and $\lambda_4=\frac{1}{4}E(X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4})$, respectively.

2.6. Rényi entropy

An entropy is a measure of how uncertain a random variable is. Rényi entropy is an extension of Shannon entropy and is expressed as

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^{\infty}[f_{EGPS}(x)]^{v}dx\right),v\neq 1,v>0. \tag{2.14}$$

Rényi entropy tends to Shannon entropy as $v\to 1$. Note that $f^{v}_{EGPS}(x)$ can be written as

$$\begin{aligned}
f^{v}_{EGPS}(x)&=\Bigg[\theta\alpha\beta g(x;\xi)\left(1-\left(1-G(x;\xi)\right)^{\alpha}\right)^{\beta-1}\left(1-G(x;\xi)\right)^{\alpha-1}\\
&\times\frac{C'\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)}{C(\theta)}\Bigg]^{v}\\
&=\frac{(\theta\alpha\beta g(x;\xi))^{v}}{C(\theta)^{v}}\left(1-\left(1-G(x;\xi)^{\alpha}\right)\right)^{(\beta-1)v}\left(1-G(x;\xi)\right)^{(\alpha-1)v}\\
&\times\left(\sum_{s=0}^{\infty}(s+1)a_{s+1}\theta^{s}\left(1-\left(1-G(x;\xi)\right)^{\alpha}\right)^{\beta s}\right)^{v}.
\end{aligned}$$

We apply the result of a power series raised to a positive integer, with $a_s=(s+1)a_{s+1}\theta^{s}$, as in equation (2.12), where $b_{s,k}=(sa_0)^{-1}\sum_{l=1}^{s}[k(l+1)-s]a_l b_{s-l,k}$, and $b_{0,k}=a_0^{k}$, see (Gradshteyn and Ryzhik (2000)), to obtain

$$\begin{aligned}
f^{v}_{EGPS}(x)&=\sum_{s=0}^{\infty}b_{s,k}\frac{(\theta\alpha\beta g(x;\xi))^{v}}{C(\theta)^{v}}\left(1-G(x;\xi)\right)^{(\alpha-1)v}\left(1-\left(1-G(x;\xi)\right)^{\alpha}\right)^{v(\beta-1)+\beta s}\\
&=\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}b_{s,k}\frac{(\theta\alpha\beta g(x;\xi))^{v}(-1)^{j}}{C(\theta)^{v}}\binom{v(\beta-1)+\beta s}{j}\left(1-G(x;\xi)\right)^{v(\alpha-1)+\alpha j}\\
&=\sum_{s=0}^{\infty}\sum_{j,k=0}^{\infty}b_{s,k}\frac{(\theta\alpha\beta)^{v}(-1)^{j+k}}{C(\theta)^{v}}\binom{v(\beta-1)+\beta s}{j}\binom{v(\alpha-1)+\alpha j}{k}g(x;\xi)^{v}G(x;\xi)^{k}.
\end{aligned}$$

Rényi entropy for the EGPS distribution is given by

$$\begin{aligned}
I_R(v)&=\frac{1}{1-v}\log\Bigg[\sum_{s=0}^{\infty}\sum_{j,k=0}^{\infty}b_{s,k}(-1)^{j+k}\binom{v(\beta-1)+\beta s}{j}\binom{v(\alpha-1)+\alpha j}{k}\\
&\times\frac{(\theta\alpha\beta)^{v}}{(C(\theta))^{v}}\int_0^{\infty}g(x,\xi)^{v}G(x;\xi)^{k}dx\Bigg],
\end{aligned}$$

for $v\neq 1,v>0$.

2.7. Maximum likelihood estimation

Let $X\sim EGPS(\theta,\alpha,\beta,\xi)$ and $\boldsymbol{\Delta}=(\theta,\alpha,\beta,\xi)^{T}$ be the parameter vector. The log-likelihood $\ell=\ell(\boldsymbol{\Delta})$ based on a random sample of size n is given by

$$\begin{aligned}
\ell(\boldsymbol{\Delta})&=n\ln(\theta\alpha\beta)+\sum_{i=1}^{n}\ln(g(x_i;\xi))+(\alpha-1)\sum_{i=1}^{n}\ln(1-G(x_i;\xi))\\
&+(\beta-1)\sum_{i=1}^{n}\ln\left[1-(1-G(x_i;\xi))^{\alpha}\right]-n\ln C(\theta)\\
&+\sum_{i=1}^{n}\ln\left(C'\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)\right).
\end{aligned}$$

Elements of the score vector $U=(\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\xi_k})$ are given by:

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-\frac{nC'(\theta)}{C\theta}+\frac{\left(C''\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)\right)\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}}{\left(C'\left(\theta\left(1-(1-G(x;\xi))^{\alpha}\right)^{\beta}\right)\right)},$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1 - G(x_i;\xi)) - (\beta - 1)\sum_{i=1}^{n} \frac{\ln(1 - G(x_i;\xi))(1 - G(x_i;\xi))^{\alpha}}{(1 - (1 - G(x_i;\xi)^{\alpha}))}$$
$$+ \theta\beta \sum_{i=1}^{n} \frac{\ln(1 - G(x_i;\xi))(1 - G(x_i;\xi))^{\alpha}\left(1 - (1 - G(x;\xi))^{\alpha}\right)^{\beta-1}}{(C'\left(\theta\left(1 - (1 - G(x;\xi))^{\alpha}\right)^{\beta}\right)}$$
$$\times \left(C''\left(\theta\left(1 - (1 - G(x;\xi))^{\alpha}\right)^{\beta}\right)\right),$$
$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(1 - (1 - G(x_i;\xi))^{\alpha}\right)$$
$$- \theta \sum_{i=1}^{n} \frac{(1 - (1 - G(x_i;\xi))^{\alpha})^{\beta} \ln\left(1 - (1 - G(x_i;\xi))^{\alpha}\right)\left(C''\left(\theta\left(1 - (1 - G(x;\xi))^{\alpha}\right)^{\beta}\right)\right)}{\left(C'\left(\theta\left(1 - (1 - G(x;\xi))^{\alpha}\right)^{\beta}\right)\right)},$$

and

$$\frac{\partial \ell}{\partial \xi_{\kappa}} = \sum_{i=1}^{n} \frac{\partial g(x_i;\xi)}{\partial \xi_{\kappa}} \frac{1}{g(x_i;\xi)} - (\alpha - 1)\sum_{i=1}^{n} \frac{g(x_i;\xi)\frac{\partial G(x_i;\xi)}{\partial \xi_{\kappa}}}{(1 - G(x_i;\xi))}$$
$$+ (\beta - 1)\alpha \sum_{i=1}^{n} \frac{\left(1 - G(x_i;\xi)\right)^{\alpha-1}\frac{\partial G(x_i;\xi)}{\partial \xi_{\kappa}}}{\left(1 - (1 - G(x_i;\xi))^{\alpha}\right)}$$
$$+ \theta\beta\alpha \sum_{i=1}^{n} \frac{(1 - (1 - G(x_i;\xi))^{\alpha})^{\beta-1}(1 - G(x_i;\xi))^{\alpha-1})\frac{\partial G(x_i;\xi)}{\partial \xi_{\kappa}}}{\left(C'\left(\theta\left(1 - (1 - G(x;\xi))^{\alpha}\right)^{\beta}\right)\right)}$$
$$\times \left(C''\left(\theta\left(1 - (1 - G(x;\xi))^{\alpha}\right)^{\beta}\right)\right),$$

respectively.

3. Exponentiated generalized logarithmic distribution

Now we consider the exponentiated generalized logarithmic (EGL) class of distributions which is a sub-class of the EGPS class of distributions. It was obtained by mixing the EG class of distributions and the logarithmic distribution. Let the random variable X denote the lifetime of a system defined by $X = \max(X_1, X_2, ..., X_N)$, where we assume that the distribution of each X_i is identically and independently distributed following the EG distribution with cdf and pdf given in equations (2.1) and (2.2). Let the random variable N follow the logarithmic distribution with pmf given as:

$$P(N = n) = \left[\frac{p^n}{-n\ln(1 - p)}\right], \quad n = 1, 2, ..., \text{ and } 0 < p < 1. \tag{3.1}$$

Then, the marginal cdf of X is given by

$$\begin{aligned} F_{EGL}(x) &= \sum_{n=1}^{\infty} P(X \le x | N = n)P(N = n) \\ &= \sum_{n=1}^{\infty} \left[[F(x)]^n\right]\left[\frac{p^n}{-n\ln(1 - p)}\right] \\ &= \frac{\ln\left[1 - p\left[1 - (1 - G(x;\xi))^{\alpha}\right]^{\beta}\right]}{\ln(1 - p)}. \end{aligned} \tag{3.2}$$

The (pdf) of the EGL distribution is given by

$$f_{EGL}(x; p, \alpha, \beta, \xi) = \frac{p\alpha\beta[1 - G(x;\xi)]^{\alpha-1}[1 - (1 - G(x;\xi))^{\alpha}]^{\beta-1} g(x;\xi)}{\ln(1 - p)\left[p\left[1 - (1 - G(x;\xi))^{\alpha}\right]^{\beta} - 1\right]}, \tag{3.3}$$

for $\alpha, \beta > 0$, $0 < p < 1$ and the parameter vector ξ.

The sub-models of EGL distribution considered in this study are Exponentiated Generalized Exponential Logarithmic (EGEL), Exponentiated Generalized Gumbel Logarithmic (EGGL), Exponentiated Generalized Uniform Logarithmic (EGUL), and Exponentiated Generalized Fréchet Logarithmic (EGFL) distributions.

3.1. Quantile function

The EGL quantile function is obtainable through the inversion of $F_{EGL}(x) = u$, $0 \le u \le 1$, where

$$F_{EGL}(x) = \frac{\ln\left[1 - p\left[1 - (1 - G(x;\xi))^{\alpha}\right]^{\beta}\right]}{\ln(1 - p)}.$$

Then solve,

$$\beta\ln[1 - (1 - G(x;\xi)^{\alpha}] = \ln\left(\frac{1 - (1 - p)^u}{p}\right),$$

Table 3.1. Table of Quantiles for Selected Parameters of EGEL.

u	(0.9,1,0.2,0.1)	(0.5,5,1,0.5)	(0.7,0.1,0.5,0.3)	(0.4,0.5,2,0.9)	(0.2,3,0.2,0.5)
0.1	0.0062	0.0575	0.8870	0.9670	0.0001
0.2	0.1166	0.1198	3.2706	1.5083	0.0003
0.3	0.5371	0.1883	6.9236	2.0140	0.0024
0.4	1.4361	0.2649	11.7995	2.5301	0.0095
0.5	2.9186	0.3525	18.0119	3.0866	0.0279
0.6	5.0867	0.4564	25.8864	3.7184	0.0677
0.7	8.1322	0.5859	36.1443	4.4821	0.1468
0.8	12.5399	0.7623	50.5087	5.4985	0.3027
0.9	19.9653	1.0537	74.6035	7.1469	0.6506

that is,

$$G(x;\xi) = 1 - \left[1 - \left(\frac{1-(1-p)^u}{p}\right)^{1/\beta}\right]^{1/\alpha}.$$

Consequently,

$$x = G^{-1}\left[1 - \left(1 - \left(\frac{1-(1-p)^u}{p}\right)^{1/\beta}\right)^{1/\alpha}\right]. \tag{3.4}$$

Therefore, random numbers for the parameters of the model can be generated using equation (3.4), when the function $G(x;\xi)$ is specified. Consider a special case where $G(x;\xi)$ is exponential distribution. Table 3.1 shows the quantiles for selected parameter values of the EGEL distribution.

3.2. Expansion of density

We apply the series expansions $(1-y)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\Gamma(k)} y^j$, $|y| < 1, k > 0$, and $(1-z)^{\beta-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\beta}{k!\Gamma(\beta-k)} z^k$ which is valid for $\beta > 0$ and $|z| < 1$. We can rewrite the EGL density function as

$$\begin{aligned}
f_{EGL}(x) &= \frac{p\alpha\beta[1-G(x;\xi)]^{\alpha-1}[1-(1-G(x;\xi))^{\alpha}]^{\beta-1} g(x;\xi)}{\ln(1-p)\left[p\left[1-(1-G(x;\xi))^{\alpha}\right]^{\beta} - 1\right]} \\
&= \frac{-\alpha\beta}{\ln(1-p)} \sum_{i=0}^{\infty} p^{i+1}(1-(1-G(x;\xi))^{\alpha})^{\beta(i+1)-1}(1-G(x;\xi))^{\alpha-1} g(x;\xi) \\
&= \sum_{i=0}^{\infty} \frac{-\alpha\beta(i+1)p^{i+1}}{\ln(1-p)(i+1)} (1-G(x;\xi))^{\alpha-1}[1-(1-G(x;\xi))^{\alpha}]^{\beta(i+1)-1} g(x;\xi) \\
&= \sum_{i=0}^{\infty} \omega(i,p) f_{EG}(x;\alpha,\beta(i+1),\xi),
\end{aligned} \tag{3.5}$$

where

$$f_{EG}(x;\alpha,\beta(i+1),\xi) = \alpha\beta(i+1)(1-G(x;\xi))^{\alpha-1}[1-(1-G(x;\xi))^{\alpha}]^{\beta(i+1)-1} g(x;\xi)$$

for $\alpha > 0, \beta(i+1) > 0$, and the parameter vector $\xi > 0$, is the exponentiated generalized (EG) class of distributions and $\omega(i,p) = \{(-1)p^{i+1}\}/\{(i+1)\ln(1-p)\}$ are the weights. It follows that the EGL class of distributions can be written as a linear combination of EG class of distributions. The mathematical and statistical properties of the EGL class of distributions can be derived from properties of the EG class of distributions.

3.3. Hazard and reverse hazard functions

The hazard and reverse hazard functions of the EGL distribution are expressed as

$$h_F(x) = \frac{p\alpha\beta[1-G(x;\xi)]^{\alpha-1}[1-(1-G(x;\xi))^{\alpha}]^{\beta-1} g(x;\xi)}{\left[p[1-(1-G(x;\xi))^{\alpha}]^{\beta} - 1\right]\left[\ln(1-p) - \ln[1-p(1-(1-G(x;\xi))^{\alpha})^{\beta}]\right]}$$

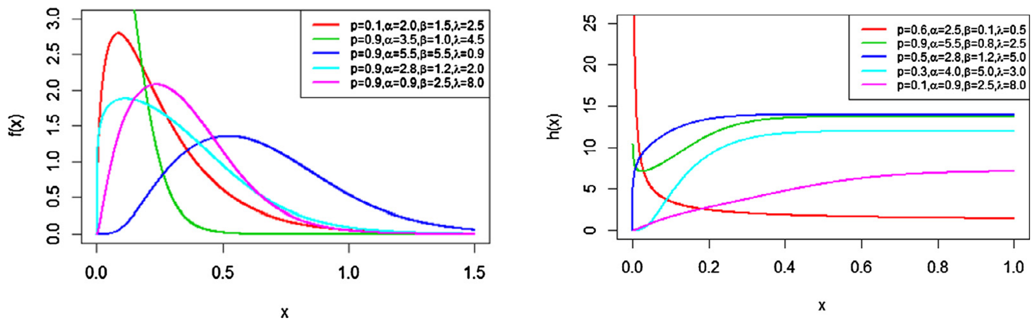
and

$$\tau_F(x) = \frac{p\alpha\beta[1-G(x;\xi)]^{\alpha-1}[1-(1-G(x;\xi))^{\alpha}]^{\beta-1} g(x;\xi)}{\left[p[1-(1-G(x;\xi))^{\alpha}]^{\beta} - 1\right]\ln\left[1-p[1-(1-G(x;\xi))^{\alpha}]^{\beta}\right]}$$

for $0 < p < 1$, $\alpha > 0, \beta > 0$ and parameter vector ξ, respectively. Plots of the hazard function can be readily obtained for specified baseline cdf $G(x;\xi)$.

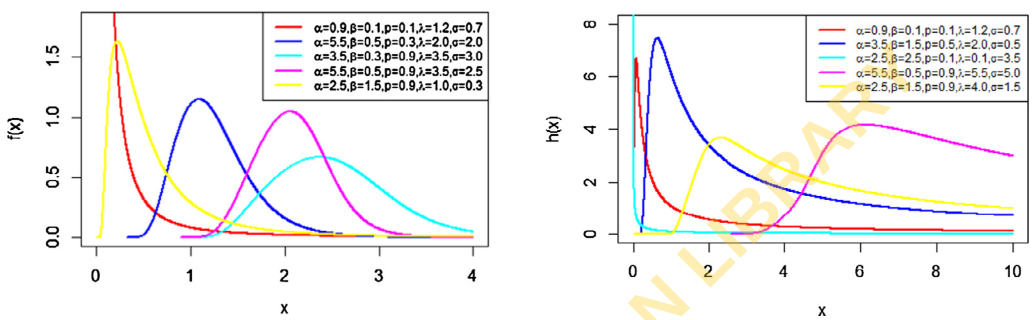
3.4. Special models

This section shows some special models of the EGL distributions.



(a) Plot of EGEL density function (b) Plot of EGEL hazard function

Fig. 3.1. Plots of EGEL Density and Hazard Function.



(a) Plot of EGFL density function (b) Plot of EGFL hazard function

Fig. 3.2. Plots of EGFL Density and Hazard Function.

3.4.1. Exponentiated generalized exponential logarithmic distribution

The cdf of the exponential distribution for $x > 0$, $\lambda > 0$, is $G(x) = 1 - e^{-\lambda x}$ and pdf given as $g(x) = \lambda e^{-\lambda x}$. Then we define the exponentiated generalized exponential logarithmic (EGEL) cdf by

$$F_{EGEL}(x) = \frac{\ln\left[1 - p\left[1 - e^{-\alpha\lambda x}\right]^{\beta}\right]}{\ln(1-p)}. \tag{3.6}$$

The corresponding pdf of the EGEL distribution is given by

$$f_{EGEL}(x) = \frac{p\alpha\beta\lambda e^{-\alpha\lambda x}\left[1 - e^{-\alpha\lambda x}\right]^{\beta-1}}{\ln(1-p)\left[p\left(1 - e^{-\alpha\lambda x}\right)^{\beta} - 1\right]}, \tag{3.7}$$

for $\alpha > 0, 0 < p < 1$, $\beta > 0$ and $\lambda > 0$. Plots of the EGEL density function and hazard function for selected parameter values are given in Fig. 3.1a and 3.1b, respectively. The EGEL pdf plot shows that the pdf can be unimodal, decreasing and right skewed depending on selected parameter values. While the plot of the EGEL hazard function shows that it can be decreasing, bathtub shaped or increasing for various selected parameter values.

3.4.2. Exponentiated generalized Fréchet logarithmic distribution

The cdf of the Fréchet distribution for $x > 0$ is $G(x) = e^{-(\frac{\sigma}{x})^{\lambda}}$, where $\lambda > 0$, and $\sigma > 0$. The exponentiated generalized Fréchet logarithmic (EGFL) cdf is defined by

$$F_{EGFL}(x) = \frac{\ln\left[1 - p\left[1 - (1 - e^{-(\frac{\sigma}{x})^{\lambda}})^{\alpha}\right]^{\beta}\right]}{\ln(1-p)}. \tag{3.8}$$

The corresponding pdf of the EGFL distribution is expressed as

$$f_{EGFL}(x) = \frac{p\alpha\beta\lambda\sigma^{\lambda}x^{(-\lambda+1)}e^{-(\frac{\sigma}{x})^{\lambda}}\left[1 - e^{-(\frac{\sigma}{x})^{\lambda}}\right]^{\alpha-1}\left[1 - (1 - e^{-(\frac{\sigma}{x})^{\lambda}})^{\alpha}\right]^{\beta-1}}{\ln(1-p)\left[p\left[1 - (1 - e^{-(\frac{\sigma}{x})^{\lambda}})^{\alpha}\right]^{\beta} - 1\right]}, \tag{3.9}$$

for $\alpha > 0, 0 < p < 1$, $\beta > 0$, $\sigma > 0$ and $\lambda > 0$. Plots of the EGFL density function and hazard function for selected parameter values are shown in Fig. 3.2a and 3.2b, respectively. The EGFL pdf plot shows that the pdf can be unimodal, symmetric bell-shaped, decreasing and right skewed depending on selected parameter values. While the plot of the EGFL hazard function shows that it can be decreasing, upside down or bathtub shaped for different selected parameter values.

3.4.3. Exponentiated generalized uniform logarithmic distribution

The cdf of the Uniform distribution is $G(x) = \frac{x}{\theta}$ and pdf given as $g(x) = \frac{1}{\theta}$ for $0 < x < \theta$. Then we define the exponentiated generalized uniform logarithmic (EGUL) cdf by

$$F_{EGUL}(x) = \frac{\ln\left[1 - p\left[1 - (1 - \frac{x}{\theta})^{\alpha}\right]^{\beta}\right]}{\ln(1-p)}. \tag{3.10}$$

The corresponding pdf of the EGUL distribution is given by

$$f_{EGUL}(x) = \frac{p\alpha\beta\theta^{-1}[1 - \frac{x}{\theta}]^{\alpha-1}[1 - (1 - \frac{x}{\theta})^{\alpha}]^{\beta-1}}{\ln(1-p)\left[p\left[1 - (1 - \frac{x}{\theta})^{\alpha}\right]^{\beta} - 1\right]}, \tag{3.11}$$

for $\alpha > 0, 0 < p < 1$, $\beta > 0$, and $0 < x < \theta$.

3.4.4. Exponentiated generalized Gumbel logarithmic distribution

The cdf of the Gumbel distribution for $x > 0$ is $G(x) = e^{-e^{-(\frac{x-\mu}{\sigma})}}$, where $\lambda > 0$, and $\sigma > 0$. Then we define the exponentiated generalized Gumbel logarithmic (EGGL) cdf by

$$F_{EGGL}(x) = \frac{\ln\left[1 - p\left[1 - (1 - e^{-e^{-(\frac{x-\mu}{\sigma})}})^{\alpha}\right]^{\beta}\right]}{\ln(1-p)}. \tag{3.12}$$

The corresponding pdf of the EGGL distribution is

$$f_{EGGL}(x) = \frac{p\alpha\beta\sigma^{-1}e^{-(\frac{x-\mu}{\sigma})}e^{-e^{-(\frac{x-\mu}{\sigma})}}[1 - e^{-e^{-(\frac{x-\mu}{\sigma})}}]^{\alpha-1}[1 - (1 - e^{-e^{-(\frac{x-\mu}{\sigma})}})^{\alpha}]^{\beta-1}}{\ln(1-p)\left[p\left[1 - (1 - e^{-e^{-(\frac{x-\mu}{\sigma})}})^{\alpha}\right]^{\beta} - 1\right]}, \tag{3.13}$$

where $\alpha > 0, 0 < p < 1$, $\beta > 0$ $x > 0$, $\mu \in \mathbb{R}$ is a location parameter, $\lambda > 0$ and $\sigma > 0$.

3.5. Moments

The rth moment of the EGL class of distributions is given by

$$E(X^r) = \sum_{i=0}^{\infty} \omega(i,p) \int_0^{\infty} x^r f_{EG}(x; \alpha, \beta(i+1), \xi) dx = \sum_{i=0}^{\infty} \omega(i,p) E(Y^r), \tag{3.14}$$

where $Y \sim EG(\alpha, \beta(i+1), \xi)$. Note that

$$\begin{aligned} E(Y^r) &= \int_0^{\infty} y^r f_{EG}(y; \alpha, \beta(i+1), \xi) dy \\ &= \alpha\beta(i+1) \int_0^{\infty} y^r (1 - G(y;\xi))^{\alpha-1} \left[1 - (1 - G(y;\xi))^{\alpha}\right]^{\beta(i+1)-1} g(y;\xi) dy \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k}\Gamma(\beta(i+1))\Gamma(\alpha(j+1))\alpha\beta(i+1)}{j!k!\Gamma(\beta(i+1)-j)\Gamma(\alpha(j+1)-k)} \\ &\times \int_0^{\infty} y^r G(y;\xi)^k g(y;\xi) dy. \end{aligned} \tag{3.15}$$

Thus, the moments of any EG distribution can be written as an infinite weighted sum of probability weighted moments (PWMs) (Greenwood et al. (1997)) of the parent distribution. The integral $\int_0^{\infty} y^r G(y;\xi)^k g(y;\xi) dy$ can be extracted from the parent quantile function $Q_G(y) = G^{-1}(y;\xi)$. Let us set $G(y;\xi) = u$, then we obtain

$$\int_0^{\infty} y^r G(y;\xi)^k g(y;\xi) dy = \int_0^1 Q_G(u)^r u^k du, \tag{3.16}$$

where the integral is now calculated over $(0,1)$. The moment generating function of the EGL class of distribution is given by

$$E(e^{tX}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(X^i),$$

where $E(X^r)$ is given by the equation (3.14). Note also that the rth moment of the EGL class of distributions can be directly obtained as follows:

$$\begin{aligned} E(X^r) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega(i,p) \frac{(-1)^{j+k}\Gamma(\beta(i+1))\Gamma(\alpha(j+1))\alpha\beta(i+1)}{j!k!\Gamma(\beta(i+1)-j)\Gamma(\alpha(j+1)-k)} \\ &\times \int_0^{\infty} x^r G(x;\xi)^k g(x;\xi) dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega(i,p) \frac{(-1)^{j+k}\Gamma(\beta(i+1))\Gamma(\alpha(j+1))\alpha\beta(i+1)}{j!k!\Gamma(\beta(i+1)-j)\Gamma(\alpha(j+1)-k)} \end{aligned}$$

Table 3.2. Table of Moments for Selected Parameters of EGEL Distribution.

	(0.9,4,0.5,0.5)	(0.5,5,1,0.5)	(0.7,6,0.5,0.3)	(0.4,2.5,0.9,0.7)	(0.2,3,0.2,0.5)
$E(X)$	0.6071	0.4746	0.4940	0.6083	0.2099
$E(X^2)$	0.7361	0.4162	0.5645	0.7272	0.2272
$E(X^3)$	1.2464	0.5247	0.9645	1.2786	0.4220
$E(X^4)$	2.6785	0.8617	2.1762	2.9624	1.0910
$E(X^5)$	6.9773	1.7472	6.0963	8.5240	3.5863
SD	0.6062	0.4370	0.5661	0.5977	0.4280
CV	0.9984	0.9206	1.1458	0.9826	2.0390
CS	1.5864	1.7482	2.0345	1.8817	3.7950
CK	6.4571	7.5689	8.9422	8.2690	23.5813

$$\times \int_0^1 Q_G(u)^r u^k du. \tag{3.17}$$

3.6. Conditional moments

The rth conditional moment for the EGL class of distributions is given by

$$\begin{aligned}
E(X^r|X>t) &= \frac{1}{F_{EGL}(t)}\int_t^\infty x^r f_{EGL}(x)dx \\
&= \frac{1}{F_{EGL}(t)}\int_t^\infty x^r \frac{p\alpha\beta[1-G(x;\xi)]^{\alpha-1}[1-(1-G(x;\xi))^\alpha]^{\beta-1}g(x;\xi)}{\ln(1-p)\left[p\left[1-(1-G(x;\xi))^\alpha\right]^\beta-1\right]}dx \\
&= \frac{-\alpha\beta}{\ln(1-p)F_{EGL}(t)}\sum_{i=1}^\infty p^{i+1} \\
&\times \int_t^\infty x^r(1-(1-G(x;\xi))^\alpha)^{\beta(i+1)-1}(1-G(x;\xi))^{\alpha-1}g(x;\xi)dx \\
&= \frac{-\alpha\beta}{\ln(1-p)F_{EGL}(t)}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \frac{(-1)^{k+j}p^{i+1}\Gamma(\beta(i+1)\Gamma(\alpha(j+1)}{j!k!\Gamma(\beta(i+1)-j)\Gamma(\alpha(j+1)-k)} \\
&\times \int_t^\infty x^r g(x;\xi)[G(x;\xi)]^k dx.
\end{aligned} \tag{3.18}$$

Table 3.2 shows the first 5 moments alongside the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the EGEL distribution for some selected values of the model parameters. These values were determined numerically using R.

3.7. Order statistics and Rényi entropy

The order statistic and Rényi entropy for the EGL distribution are discussed in this section.

3.7.1. Order statistics

The pdf of the ith order statistic from the EGL pdf $f_{EGL}(x)$ is

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!f_{EGL}(x)}{(i-1)!(n-i)!}[F_{EGL}(x)]^{i-1}[1-F_{EGL}(x)]^{n-i} \\
&= \frac{n!f_{EGL}(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F_{EGL}(x)]^{j+i-1}
\end{aligned}$$

by using the binomial expansion $[1-F_{EGL}(x)]^{n-i} = \sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m[F_{EGL}(x)]^m$, so that

$$\begin{aligned}
g_{i:n}(x) &= \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\binom{n-i}{m}\frac{(-1)^m}{m+i}(m+i)[F_{EGL}(x)]^{m+i-1}f_{EGL}(x) \\
&= \sum_{m=0}^{n-i} w_{i,m} f_{m+i}(x),
\end{aligned}$$

where $f_{m+i}(x)$ is the pdf of the EGL distribution with parameters α,β,p and $(m+i)$; $B(.,.)$ is the beta function and the weights $w_{i,m}$ are expressed as

$$w_{i,m} = \frac{1}{B(i,n-i+1)}\frac{(-1)^m}{m+i}\binom{n-i}{m} = (-1)^m\binom{m+i-1}{m}\binom{n}{m+i}.$$

The tth moment of the ith order statistics from the EGL distribution is derivable using the result of Barakat and Abdelkader (2004) as follows:

$$E(X^t_{i:n}) = t \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i}\binom{n}{j} \int_0^\infty x^{t-1}[1-F_{EGL}(x)]^j dx. \tag{3.19}$$

Note that

$$\int_0^\infty x^{t-1}[1-F_{EGL}(x)]^j dx = \sum_{k=0}^\infty \frac{(-1)^{2k}}{(\ln(1-p))^k}\binom{j}{k} \times \int_0^\infty x^{t-1}\left(\sum_{s=0}^\infty \frac{p^{s+1}(1-(1-G(x;\xi))^\alpha)^{\beta(s+1)}}{s+1}\right)^k dx.$$

We apply the result of a power series raised to a positive integer, with $a_s = (1-p)^s(s+1)^{-1}$, that is,

$$\left(\sum_{s=0}^\infty a_s y^s\right)^k = \sum_{s=0}^\infty b_{s,k} y^s, \tag{3.20}$$

where $b_{s,k} = (sa_0)^{-1}\sum_{l=1}^s [k(l+1)-s]a_l b_{s-l,k}$, and $b_{0,k} = a_0^k$, (Gradshteyn and Ryzhik (2000)), to obtain

$$\begin{aligned}\int_0^\infty x^{t-1}[\overline{F}_{EGL}(x)]^j dx &= \sum_{k=0}^\infty \frac{(-1)^{2k}p^k}{(\ln(1-p))^k}\binom{j}{k}\int_0^\infty x^{t-1}(1-(1-G(x;\xi))^\alpha)^{\beta k} \\ &\times \left(\sum_{s=0}^\infty \frac{p^s}{s+1}(1-(1-G(x;\xi))^\alpha)^{\beta s}\right)^k dx \\ &= \sum_{k=0}^\infty \sum_{s,w,z=0}^\infty b_{s,k}\frac{(-1)^{2k+w+z}p^k}{(\ln(1-p))^k}\binom{j}{k}\binom{\beta(k+s)}{w}\binom{\alpha w}{z} \\ &\times \int_0^\infty x^{t-1}(G(x;\xi))^z dx.\end{aligned}$$

3.8. Rényi entropy

The Rényi entropy tends to Shannon entropy as $v \to 1$. Note that $f^v_{EGL}(x)$ can be written as

$$\begin{aligned}f^v_{EGL}(x) &= \left[\frac{p\alpha\beta(1-G(x;\xi))^{\alpha-1}[1-(1-G(x;\xi))^\alpha]^{\beta-1}g(x;\xi)}{[p[1-(1-G(x;\xi))^\alpha]^\beta - 1]\ln(1-p)}\right]^v \\ &= \sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(-1)^{v+k+m}\Gamma(v+j)\Gamma(v(\beta-1)+1)\Gamma(\alpha(v+k)-v+1)}{j!k!m!\Gamma(v)\Gamma(v(\beta-1)-k+1)\Gamma(\alpha(v+k)-v-m+1)} \\ &\times \frac{(p\alpha\beta)^v}{(\ln(1-p))^v} G(x;\xi)^m g(x)^v.\end{aligned}$$

Consequently, Rényi entropy for the EGL distribution is given by

$$\begin{aligned}I_R(v) = \frac{1}{1-v}\log\Bigg[&\sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(-1)^{v+k+m}\Gamma(v+j)\Gamma(v(\beta-1)+1)\Gamma(\alpha(v+k)-v+1)}{j!k!m!\Gamma(v)\Gamma(v(\beta-1)-k+1)\Gamma(\alpha(v+k)-v-m+1)} \\ &\times \frac{(p\alpha\beta)^v}{(\ln(1-p))^v}\int_0^\infty G(x;\xi)^m g(x;\xi)^v dx\Bigg].\end{aligned}$$

3.9. Maximum likelihood estimation

Let $X \sim EGL(p,\alpha,\beta,\xi)$ and $\mathbf{\Delta} = (p,\alpha,\beta,\xi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\mathbf{\Delta})$ based on a random sample of size n can be expressed as

$$\begin{aligned}\ell &= n\ln(p\alpha\beta) - n\ln(\ln(1-p)) + (\alpha-1)\sum_{i=1}^n \ln(1-G(x_i;\xi)) \\ &+ (\beta-1)\sum_{i=1}^n \ln\left[1-(1-G(x_i;\xi))^\alpha\right] + \sum_{i=1}^n \ln(g(x_i;\xi)) \\ &- \sum_{i=1}^n \ln\left[p\left(1-(1-G(x_i;\xi))^\alpha\right)^\beta - 1\right].\end{aligned}$$

Table 4.1. EGEL Distribution: Monte Carlo Simulation Results.

Parameter	Sample Size	(0.2, 1.0, 1.0, 0.5)			(0.5, 2.0, 1.0, 0.5)			(0.2, 2.0, 2.0, 0.6)		
		Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.5653	0.4548	0.3653	0.6513	0.2783	0.1513	0.6072	0.4835	0.4072
	50	0.5133	0.4042	0.3133	0.5598	0.2583	0.0598	0.5383	0.4316	0.3383
	100	0.4589	0.3515	0.2589	0.5717	0.2481	0.0717	0.5245	0.4088	0.3245
	200	0.4287	0.3168	0.2287	0.5043	0.2419	0.0043	0.4828	0.3680	0.2828
	400	0.3500	0.2436	0.1500	0.4631	0.2057	-0.0369	0.3923	0.2852	0.1923
	800	0.3037	0.1953	0.1037	0.4601	0.1800	-0.0399	0.3381	0.2213	0.1381
β	25	1.1135	0.2349	0.1135	2.1620	0.3259	0.1620	2.1880	0.5026	0.1880
	50	1.0724	0.1884	0.0724	2.1495	0.2923	0.1495	2.1126	0.3498	0.1126
	100	1.0599	0.1232	0.0599	2.1274	0.2395	0.1274	2.0907	0.2747	0.0907
	200	1.0552	0.1071	0.0552	2.0904	0.1612	0.0904	2.0918	0.2356	0.0918
	400	1.0373	0.0758	0.0373	2.0549	0.1237	0.0549	2.0548	0.1718	0.0548
	800	1.0317	0.0617	0.0317	2.0368	0.0876	0.0368	2.0477	0.1285	0.0477
λ	25	1.0157	0.3348	0.0157	1.0448	0.3344	0.0448	2.2067	0.8715	0.2067
	50	0.9649	0.2430	-0.0351	1.0252	0.2824	0.0252	2.0070	0.5570	0.0070
	100	0.9701	0.1508	-0.0299	1.0021	0.1688	0.0021	1.9727	0.3399	-0.0273
	200	0.9621	0.1104	-0.0379	1.0084	0.1252	0.0084	1.9371	0.2308	-0.0629
	400	0.9773	0.0791	-0.0227	1.0123	0.0967	0.0123	1.9619	0.1679	-0.0381
	800	0.9838	0.0595	-0.0162	1.0136	0.0739	0.0136	1.9754	0.1238	-0.0246
p	25	0.5669	0.1215	0.0669	0.6037	0.1681	0.1037	0.7063	0.1803	0.1063
	50	0.5454	0.0902	0.0454	0.5492	0.1024	0.0492	0.6690	0.1349	0.0690
	100	0.5291	0.0601	0.0291	0.5390	0.0764	0.0390	0.6609	0.1078	0.0609
	200	0.5171	0.0455	0.0171	0.5167	0.0521	0.0167	0.6397	0.0861	0.0397
	400	0.5070	0.0305	0.0070	0.5021	0.0352	0.0021	0.6229	0.0606	0.0229
	800	0.5022	0.0234	0.0022	0.5006	0.0263	0.0006	0.6133	0.0430	0.0133

Elements of the score vector $U = (\frac{\partial \ell}{\partial p}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \xi})$ are given by:

$$\frac{\partial \ell}{\partial p} = \frac{n}{p} - \frac{n}{(1-p)\ln(1-p)} - \sum_{i=1}^{n} \frac{(1-(1-G(x_i;\xi))^{\alpha})^{\beta}}{p(1-(1-G(x_i;\xi))^{\alpha})^{\beta}-1},$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-G(x_i;\xi)) + (\beta-1)\sum_{i=1}^{n}\frac{(1-G(x_i;\xi))^{\alpha}\ln(1-G(x_i;\xi))}{1-(1-G(x_i;\xi))^{\alpha}}$$
$$-\sum_{i=1}^{n}\frac{p\beta(1-(1-G(x_i;\xi))^{\alpha})^{\beta-1}(1-G(x_i;\xi))^{\alpha}\ln(1-G(x_i;\xi))}{p(1-(1-G(x_i;\xi))^{\alpha})^{\beta}-1},$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln(1-(1-G(x_i;\xi))^{\alpha}) - \sum_{i=1}^{n}\frac{p(1-(1-G(x_i;\xi))^{\alpha})^{\beta}\ln(1-(1-G(x_i;\xi))^{\alpha})}{p(1-(1-G(x_i;\xi))^{\alpha})^{\beta}-1},$$

and

$$\frac{\partial \ell}{\partial \xi_{\kappa}} = (\alpha-1)\sum_{i=1}^{n}\frac{\partial G(x_i;\xi)}{\partial \xi_{\kappa}}\frac{1}{1-G(x_i;\xi)} + \alpha(\beta-1)\sum_{i=1}^{n}\frac{\partial G(x_i;\xi)}{\partial \xi_{\kappa}}\frac{(1-G(x_i;\xi))^{\alpha-1}}{1-(1-G(x_i;\xi))^{\alpha}}$$
$$+\sum_{i=1}^{n}\frac{\partial g(x_i;\xi)}{\partial \xi_{\kappa}}\frac{1}{g(x_i;\xi)} + p\beta\alpha\sum_{i=1}^{n}\frac{\partial G(x_i;\xi)}{\partial \xi_{\kappa}}\frac{(1-(1-G(x_i;\xi))^{\alpha})^{\beta-1}(1-G(x_i;\xi))^{\alpha-1}}{p(1-(1-G(x_i;\xi))^{\alpha})^{\beta}-1},$$

respectively. Solving the nonlinear system equations, $U = (\frac{\partial \ell}{\partial p}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \xi}) = \mathbf{0}$ yields the maximum likelihood estimates. These equations can be solved numerically via iterative methods such as Newton-Raphson technique using statistical software.

4. Simulation study

We examined how well the EGL distribution performed by conducting simulation study using the special case of EGEL distribution at different sample sizes (n = 25, 50, 100, 200, 400, and 800) via the R package. We first simulated $n = 1000$ samples for the true parameters values given in the Table 4.1. Table 4.1 lists the mean MLEs, biases and root mean squared errors (RMSEs) for each of the four model parameters. The bias and RMSE are estimated using

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i-\theta)^2}{N}},$$

respectively, where θ denote the parameters α, β, λ and p. From the results, we observed that as the sample size n increases, the RMSEs decays toward zero while the biases decrease as the sample size n increase for each parameter.

5. Applications

Using real data sets, we compare and illustrate the flexibility of the special cases of EGEL and EGFL distributions. The models are compared with their sub-models and some non-nested models. The EGEL distribution is compared with the exponentiated generalized exponential and exponential

Table 5.1. Parameter estimates and goodness-of-fit statistics for Recidivism failure times (in days).

Model	Estimates				Statistics									
	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$-2\log L$	AIC	$AICC$	$CAIC$	BIC	W^*	A^*	KS	$P-value$	SS
EGEL	0.9970	0.1247	0.8575	0.1158	759.95	767.95	768.67	768.67	776.40	0.0178	0.1371	0.0449	0.9997	0.0000
	(0.0023)	(0.0711)	(0.2611)	(0.0660)										
EGE	-	1.8789	1.4610	0.0031	771.72	777.72	778.14	778.14	784.05	0.1625	1.0123	0.1144	0.4018	0.1197
	-	(0.4542)	(0.2560)	(0.0006)										
E	-	-	-	0.0047	775.34	777.34	777.41	777.41	779.45	0.1548	0.9666	0.1363	0.2074	0.3324
	-	-	-	(0.0006)										
Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\lambda}$										
$BGExponential$	2.3558	0.0716	0.1160	0.0607	779.67	787.67	788.38	788.39	796.11	0.1124	0.7005	0.1562	0.1018	0.4480
	(0.9900)	(0.0102)	(0.0486)	(0.0028)										
$KwGExponential$	0.3025	0.0496	0.5577	0.0625	807.69	815.69	816.40	816.40	824.13	0.1034	0.6458	0.2883	0.0001	1.7593
	(0.0736)	(0.0100)	(0.0834)	(0.0028)										

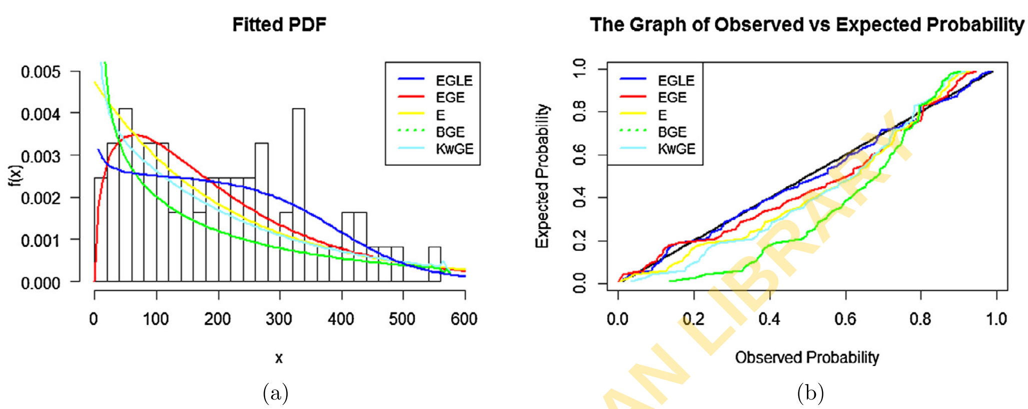


Fig. 5.1. Fitted densities and probability plots for recividism failure times data.

distribution. The EGFL distribution is compared with the exponentiated generalized Fréchet and Fréchet distribution. We also compare the EGEL and EGFL distributions with the Generalized Beta-G (BG) distribution due to Alexander et al. (2012)and Exponentiated Kumaraswamy-G (KwG) distribution proposed by Lemonte, Barreto-Souza and Cordeiro (2013).

The pdf of the generalized beta-G (BG) distribution by Alexander et al. (2012) is given below as

$$f(x) = \frac{c}{B(a,b)} g(x) G^{ac-1}(x) \left[1 - G^c(x)\right]^{b-1}, \tag{5.1}$$

while the pdf of the exponentiated Kumaraswamy G (KwG) distribution proposed by Lemonte, Barreto-Souza and Cordeiro (2013) is

$$f(x) = abc g(x) G^{a-1}(x) \left[1 - G^a(x)\right]^{b-1} \left\{1 - \left[1 - G^a(x)\right]^b\right\}^{c-1} \tag{5.2}$$

for G any valid cdf, g the corresponding pdf and $a, b, c > 0$.

The MLEs of the EGEL and EGFL parameters are computed by maximizing the objective function via the function nlm in R. The estimates (standard error), -2log-likelihood statistic, Akaike Information Criterion, ($AIC = 2p - 2\ln(L)$), Consistent Akaike Information Criterion, ($AICC = AIC + \frac{2p(p+1)}{n-p-1}$) and (Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, where $L = L(\hat{\Delta})$) are presented in Tables 5.1, and 5.3 for the sub-models and non-nested models KwG and BG distributions.

The goodness-of-fit statistics W^* and A^*, (by Chen and Balakrishnan (1985)), the Kolmogorov-Smirnov (KS) goodness-of-fit statistic and its p-value are also presented. The smaller the values of W^*, A^* and KS the better the fit and are used to identify the model that fits the data most.

The likelihood ratio (LR) test was used to compare the fitness of the EGEL and EGFL distributions with their sub-models for the real life data sets. For example, to test $\beta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{p})) - \ln(L(\tilde{\xi}, \tilde{\alpha}, 1, \tilde{p}))]$, where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\xi}$, and $\hat{p}$ are the unrestricted estimates, and $\tilde{\alpha}$, $\tilde{\xi}$, and $\tilde{p}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom. The plots of the fitted densities, the histogram and probability plots using the methods proposed by Chambers et al. (1983) are presented in Fig. 5.1 and Fig. 5.2 for all the models considered.

We plotted $F_{EGL}(x_{(j)}; \hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{p})$ against $\dfrac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data in the probability plot. A measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F_{EGL}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25}\right)\right]^2.$$

5.1. Recidivism failure times (in days) data

This data set contains 61 observed recidivism failure times (in days) revealed by correctional institutions in Columbia USA by Stollmack and Harris (1974). The failure times data were: 1, 6, 9, 29, 30, 34, 39, 41, 44, 45, 49, 56, 84, 89, 91, 100, 103, 104, 115, 119, 124, 138, 141, 146, 156, 162, 168, 183, 185, 198, 209, 217, 217, 228, 233, 238, 241, 252, 258, 271, 275, 276, 279, 282, 305, 313, 329, 331, 334, 336,336, 362, 384, 404, 408, 422, 438, 441, 465, 486, 556. Estimates of the parameters of EGEL distribution and its related sub-models, non nested models (standard error in parentheses), $-2\ln(L)$, AIC, AICC, BIC A^*, W^*, KS and its p-value are given in Table 5.1.

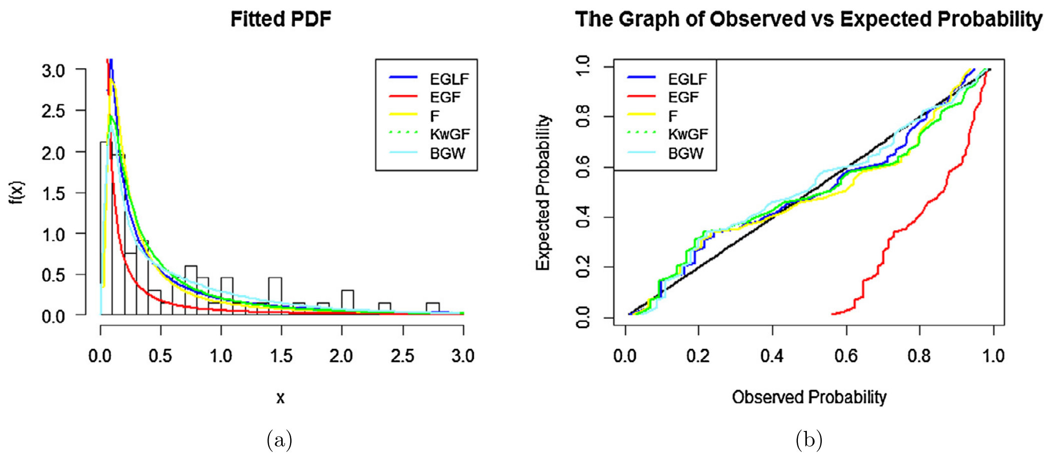


Fig. 5.2. Fitted densities and probability plots for plasma concentrations of indomethicin data.

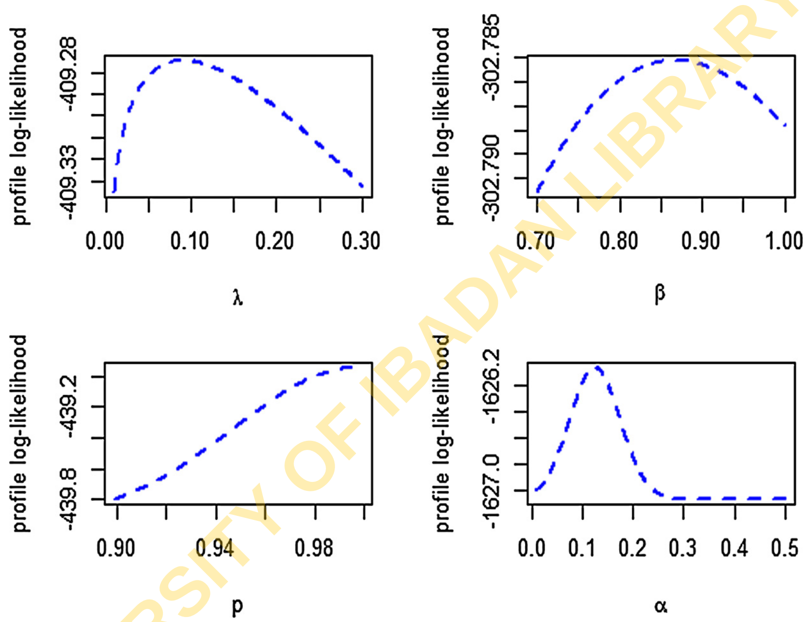


Fig. 5.3. Plots of the profile likelihood function of the parameters of the EGEL on the recividism failure times data.

Table 5.2. Plasma Concentration data.

1.50	0.94	0.78	0.48	0.37	0.19	0.12	0.11	0.08	0.07
0.05	2.03	1.63	0.71	0.70	0.64	0.36	0.32	0.20	0.25
0.12	0.08	2.72	1.49	1.16	0.80	0.80	0.39	0.22	0.12
0.11	0.08	0.08	1.85	1.39	1.02	0.89	0.59	0.40	0.16
0.11	0.10	0.07	0.07	2.05	1.04	0.81	0.39	0.30	0.23
0.13	0.11	0.08	0.10	0.06	2.31	1.44	1.03	0.84	0.64
0.42	0.24	0.17	0.13	0.10	0.09				

The LR test statistic of the hypothesis H_0: EGE against H_a: EGEL and H_0: E against H_a: EGEL, are 11.77 (p-value $= 0.0006$) and 15.39 (p-value$= 0.0015$), respectively which indicates that the tests are significant. Therefore, there are significant differences between EGEL, EGE and E distributions. Also, the AIC, AICC and BIC also showed that EGEL distribution is a better fit than the non-nested BGExponential and KwGExponential distributions for the recidivism failure times data (days). Based on the goodness-of-fit statistics W^*, A^*, KS and associated p-values, the EGEL distribution is a better fit for the recidivism failure times (days) data.

Fig. 5.3 shows the profile likelihood function of the parameters of the EGEL distribution on the recividism failure times data. Clearly, the plots are unimodal for all estimated parameters of the model.

5.2. Plasma concentration data

This data set consists of plasma concentrations of indomethicin following intravenous injection. We used the pooled data with 66 observations (Kwan et al. (1976)). The data values are given Table 5.2.

Parameter estimates of EGFL distribution and its sub-models (standard error in parentheses), $-2\ln(L)$, AIC, AICC, BIC A^*, W^*, KS and its p-value are given in Table 5.3.

Table 5.3. Parameter estimates and goodness-of-fit statistics for various models fitted for the plasma concentration data set.

Model	Estimates					Statistics									
	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\sigma}$	$-2\log L$	AIC	$AICC$	$CAIC$	BIC	W^*	A^*	KS	$P-value$	SS
EGFL	0.9969	0.6778	0.2016	2.5753	0.1336	45.83	55.83	56.83	56.83	66.77	0.1475	0.8890	0.1067	0.4400	0.1230
	(0.0022)	(0.1906)	(0.1911)	(0.5429)	(0.0543)										
EGF	-	5.5068	0.1037	0.8140	4.1373	50.80	58.80	59.45	59.45	67.56	0.1820	1.1314	0.1377	0.1635	8.7007
	-	(0.0724)	(0.0130)	(0.0028)	(0.0806)										
F	-	-	-	1.0196	0.1876	56.92	60.92	61.11	61.11	65.30	0.2482	1.5142	0.1299	0.2154	0.2441
	-	-	-	(0.0996)	(0.0239)										
$BGWeibull$	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$										
	0.1533	0.1150	17.0159	1.0342	11.1016	46.10	56.10	57.10	57.10	67.05	0.1221	0.8097	0.1275	0.2338	0.1359
	(0.0248)	(0.0174)	(0.4899)	(0.0157)	(0.0164)										
$KwGFrechet$	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\lambda}$	$\hat{\sigma}$										
	4.2191	6.3797	0.1046	0.7940	0.7307	50.75	60.75	61.75	61.75	71.70	0.1789	1.1153	0.1350	0.1801	0.2531
	(0.0038)	(0.1090)	(0.0129)	(0.0038)	(0.0038)										

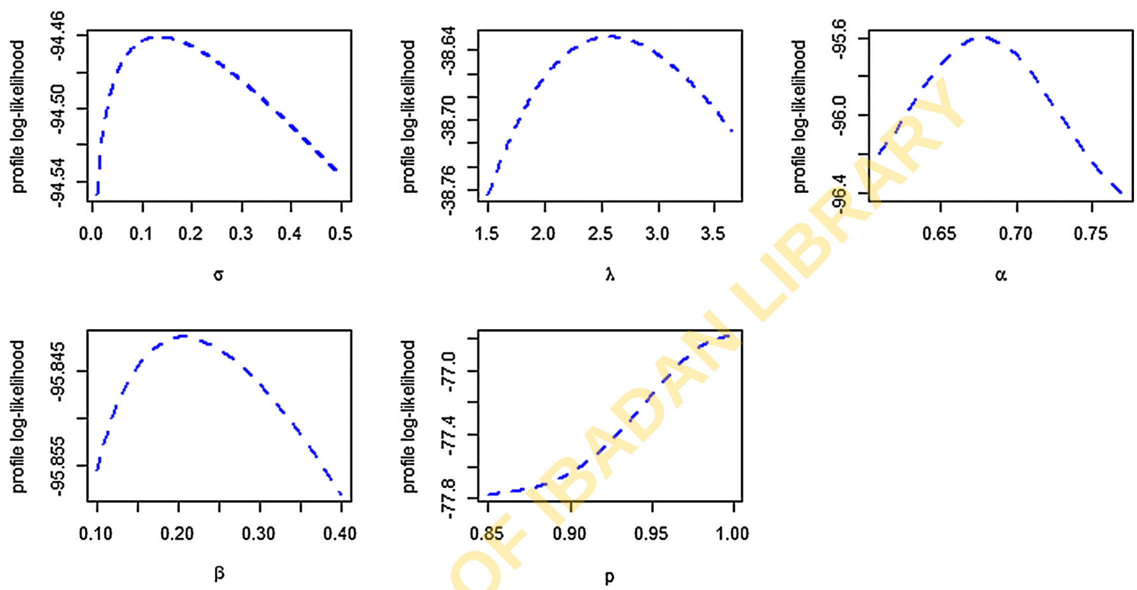


Fig. 5.4. Plots of the profile likelihood function of the parameters of the EGFL on the plasma concentrations of indomethicin data.

The LR test statistic of the hypothesis H_0: EGF against H_a: EGFL and H_0: F against H_a: EGFL are 4.97 (p-value $= 0.0258$) and 11.09 (p-value$= 0.0112$), respectively. This shows that there is significance difference between EGFL and EGF and also between EGFL and F distributions. The values of the statistics: AIC, AICC and BIC, and the goodness-of-fit statistics A^*, W^*, KS and its p-value also shows that the EGFL distribution is a better fit than the non-nested BGWeibull and KwGFréchet distributions.

Fig. 5.4 shows the profile likelihood function of the parameters of the EGFL distribution on the plasma concentrations of indomethicin data. Clearly, the plots are unimodal for all estimated parameters of the model.

6. Concluding remarks

A new generalized class of distributions called the exponentiated generalized power series (EGPS) family of distributions was developed and presented. This general family of distributions and some of its structural properties and maximum likelihood estimates are presented. Applications of the EGEL and EGFL models to real life data sets are given in order to illustrate the flexibility, applicability and usefulness of the new distribution. The distributions have a better fit than some of its sub-models and non-nested distributions for the recividism failure times data and plasma concentration data.

Declarations

Author contribution statement

B. Oluyede: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed materials, analysis tools or data; Wrote the paper.

B. Mashabe, A. Fagbamigbe, B. Makubate, D. Wanduku: Performed the experiments; Analyzed and interpreted the data; Contributed materials, analysis tools or data; Wrote the paper.

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Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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