VECTORIAL MECHANICS

MAT 242

MAT 242 - VECTORIAL MECHANICS

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Vectorial Mechanics

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Vice-Chancellor's Message

I congratulate you on being part of the historic evolution of our Centre for External Studies into a Distance Learning Centre. The reinvigorated Centre, is building on a solid tradition of nearly twenty years of service to the Nigerian community in providing higher education to those who had hitherto been unable to benefit from it.

Distance Learning requires an environment in which learners themselves actively participate in constructing their own knowledge. They need to be able to access and interpret existing knowledge and in the process, become autonomous learners.

Consequently, our major goal is to provide full multi media mode of teaching/learning in which you will use not only print but also video, audio and electronic learning materials.

To this end, we have run two intensive workshops to produce a fresh batch of course materials in order to increase substantially the number of texts available to you. The authors made great efforts to include the latest information, knowledge and skills in the different disciplines and ensure that the materials are user-friendly. It is our hope that you will put them to the best use.

Professor Olufemi A. Bamiro, FNSE
Vice-Chancellor

Foreword

The University of Ibadan Distance Learning Programme has a vision of providing lifelong education for Nigerian citizens who for a variety of reasons have opted for the Distance Learning mode. In this way, it aims at democratizing education by ensuring access and equity.

The U.I. experience in Distance Learning dates back to 1988 when the Centre for External Studies was established to cater mainly for upgrading the knowledge and skills of NCE teachers to a Bachelors degree in Education. Since then, it has gathered considerable experience in preparing and producing course materials for its programmes. The recent expansion of the programme to cover Agriculture and the need to review the existing materials have necessitated an accelerated process of course materials production. To this end, one major workshop was held in December 2006 which have resulted in a substantial increase in the number of course materials. The writing of the courses by a team of experts and rigorous peer review have ensured the maintenance of the University's high standards. The approach is not only to emphasize cognitive knowledge but also skills and humane values which are at the core of education, even in an ICT age.

The materials have had the input of experienced editors and illustrators who have ensured that they are accurate, current and learner friendly. They are specially written with distance learners in mind, since such people can often feel isolated from the community of learners. Adequate supplementary reading materials as well as other information sources are suggested in the course materials.

The Distance Learning Centre also envisages that regular students of tertiary institutions in Nigeria who are faced with a dearth of high quality textbooks will find these books very useful. We are therefore delighted to present these new titles to both our Distance Learning students and the University's regular students. We are confident that the books will be an invaluable resource to them.

We would like to thank all our authors, reviewers and production staff for the high quality of work.

Best wishes.

Professor Francis O. Egbokhare
Director

General Introduction and Course Objectives

The purpose of this course was to build on what the students have learned from physical vectors, in particular of their geometric representatives, to a study of algebraic vectors, system of line vectors and dynamics which provide the students with a precise account of this branch of Mathematics, called Vectorial Mechanics.

The objectives, upon successful completion of this course are as follows. The student should be able to:

1. have knowledge of fundamental sciences,
2. have proficiency in problem solving and analysis,
3. acquire the fundamental knowledge for the understanding of the phenomenology in vectorial mechanics,
4. develop, alone and in a team, capacities to solve exercises in the fore-mentioned subjects and the abilities to express the results in both oral and written form.

LECTURE 1

Vectors in Space

Introduction

Vectors in space are the three-dimensional analog of vectors in the plane and are subject to the same rules of addition, subtraction, and scalar multiplication that govern vectors in the plane. We shall discuss length or magnitude and direction of vectors.

Objective

At the end of this lecture, you should be able to:

- determine the lengths of any given vectors and in space,
- write down equation for the sphere of given radius with given center and
- find the direction of any given vectors in space

Pre-Test (see Post-Test)

Definition (Basic Unit Vectors)

The vectors from the origin to the points whose cartesian coordinates are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are the basic unit vectors.

We denote them by $\vec{i}, \vec{j}$ and $\vec{k}$, and write the vector from the origin $O(0,0,0)$ to the point $P(x,y,z)$ as

$$\vec{V} = \overrightarrow{OP} = \vec{i}x + \vec{j}y + \vec{k}z \tag{1}$$

If $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are two points in space (see Figure 1 below)

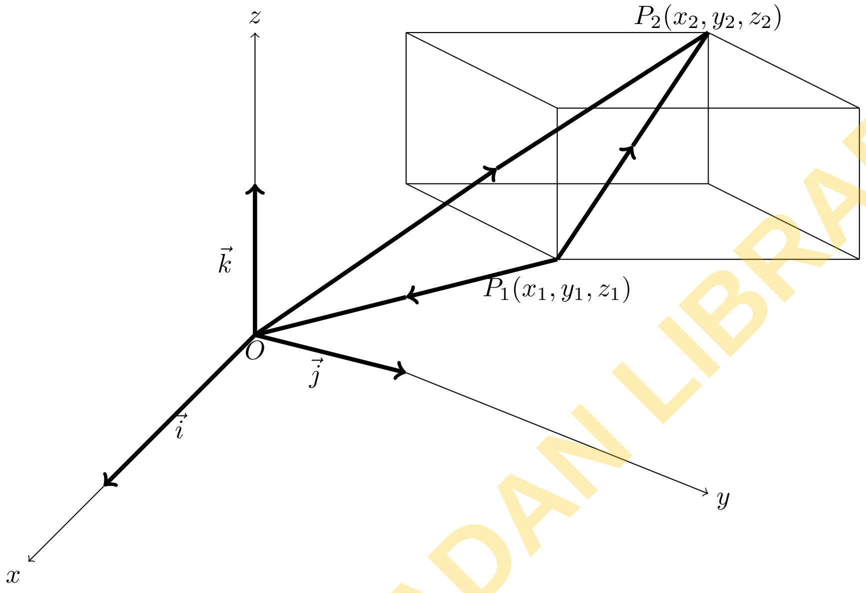


Figure 1: $P_1\vec{P}_2 = P_1\vec{O} + O\vec{P}_2$

then the vector from P_1 to P_2 is the vector sum

$$\overrightarrow{P_1P_2} = \overrightarrow{P_1O} + \overrightarrow{OP_2}.$$

Since

$$\overrightarrow{P_1O} = -\overrightarrow{OP_1}$$

this is the same as

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

or

$$\overrightarrow{P_1P_2} = \vec{i}(x_2 - x_1) + \vec{j}(y_2 - y_1) + \vec{k}(z_2 - z_1) \tag{2}$$

The length of any vector

$$\vec{P} = a\vec{i} + b\vec{j} + c\vec{k}$$

is readily determined by applying the theorem of Pythagoras twice. In the right triangle ABC (see Figure 2 below)

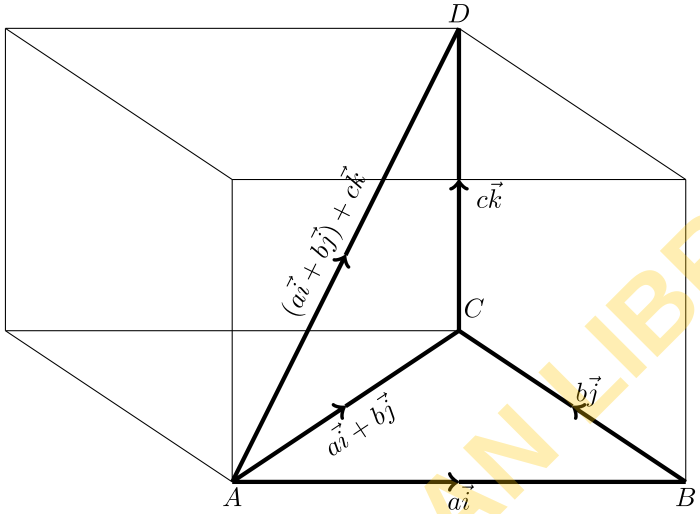


Figure 2: $|\vec{AD}|$ can be determined from the right triangles ABC and ACD

$$|\overrightarrow{AC}| = |a\vec{i} + b\vec{j}| = \sqrt{a^2 + b^2},$$

and in the right triangle ACD,

$$|\overrightarrow{AD}| = \sqrt{|\overrightarrow{AC}|^2 + |\overrightarrow{CD}|^2} = \sqrt{(a^2 + b^2) + c^2}.$$

That is,

$$|a\vec{i} + b\vec{j} + c\vec{k}| = \sqrt{a^2 + b^2 + c^2} \tag{3}$$

If we apply this result to the vector $\overrightarrow{P_1P_2}$ of Eqn (2), we obtain a formula for the distances between two points:

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{4}$$

Equation (4) may be used to determine an equation for the sphere of radius a with center at $P_0(x_0, y_0, z_0)$ (see Figure 3 below)

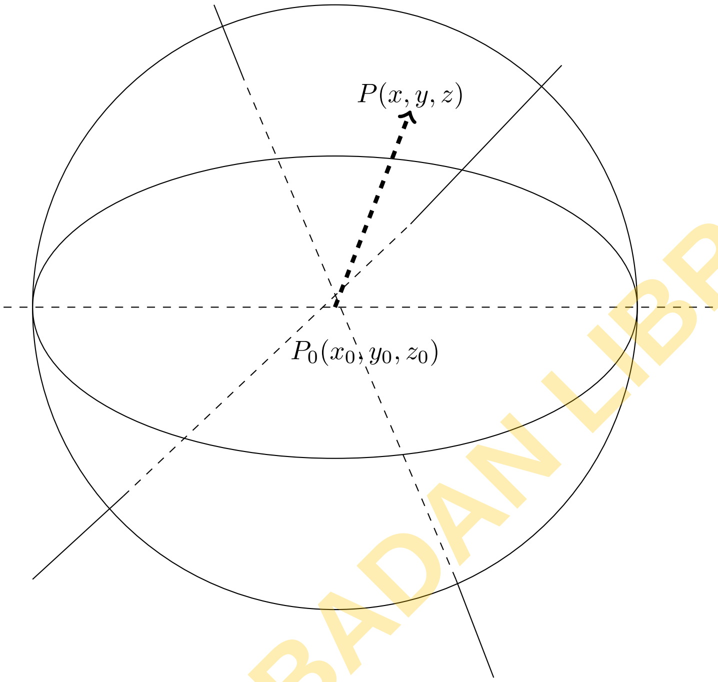


Figure 3: The sphere $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$

The point P is on the sphere if and only if

$$|\overrightarrow{P_0P}| = a,$$

or

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2 \tag{5}$$

Example: Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 2x - 4y = 0.$$

Solution: Complete the squares in the given equation to obtain

$$x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 = 1 + 4$$

$$(x + 1)^2 + (y - 2)^2 + z^2 = 5.$$

Comparison with Eqn (5) shows that $x_0 = -1$, $y_0 = 2$, $z_0 = 0$ and $a = \sqrt{5}$. The center is $(-1, 2, 0)$ and the radius is $\sqrt{5}$.

Direction

For any non-zero vector $\vec{v}$, we obtain a unit vector called the direction of $\vec{v}$ by dividing $\vec{v}$ by its own length:

$$\text{Direction of } \vec{v} = \frac{\vec{v}}{|\vec{v}|} \tag{6}$$

Example 2: If

$$\vec{v} = 2\vec{i} - 3\vec{j} + 7\vec{k},$$

then its length is $\sqrt{4+9+49} = \sqrt{62}$, and

$$\text{Direction of } (2\vec{i} - 3\vec{j} + 7\vec{k}) = \frac{2\vec{i} - 3\vec{j} + 7\vec{k}}{\sqrt{62}}$$

Summary

We have covered the following in this lecture:

1. A formula for the distance between two points in space is determined.
2. The equation for the sphere of given radius with given center is determined, and
3. The formula for the direction of any given vector is defined.

Post-Test

1. Find the centers and radii of the spheres
   (a) $x^2 + y^2 + z^2 + 4x - 4z = 0$
   (b) $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9$
2. Find the distance between the point $P(x, y, z)$ and (a) the x-axis, (b) the y-axis, (c) the z-axis, (d) the xy-plane.
3. The distance from $P(x, y, z)$ to the origin is d_1 and the distance from P to $A(0, 0, 3)$ is d_2.
   Write an equation for the coordinates of P if (a) $d_1 = 2d_2$, (b) $d_1 + d_2 = 6$, (c) $|d_1 - d_2| = 2$.
4. Find the lengths of the following vectors:
   (a) $2\vec{i} + \vec{j} - 2\vec{k}$,
   (b) $3\vec{i} - 6\vec{j} + 2\vec{k}$
   (c) $\vec{i} + 4\vec{j} - 8\vec{k}$,
   (d) $9\vec{i} - 2\vec{j} + 6\vec{k}$
5. Find the direction of $4\vec{i} + 3\vec{j} + 12\vec{k}$

6. Find the vector from the origin O to the point of intersection of the medians of the triangle whose vertices are the three points.
   $A(1, -1, 2)$, $B(2, 1, 3)$, $C(-1, 2, -1)$.

Supplementary Reading

1. M.R. Spiegel, Theory and Problems of Vector Analysis, Schaum Publishing Co., New York (1959).

2. E.A. Milne, Vectorial Mechanics (New York Interscience Publishers Inc., 1948). pp. xiii 382, ASIN:B000EGLGX.

3. F.P. Beer and E.R. Johnston, Vector Mechanics for Engineers (Statics). McGraw-Hill, 3rd Edition (1999).

4. R.A. Serway and J.W. Jewett, Physics for Scientists and Engineers (6th ed.). Brooks/Cole. ISBN 0-534-40842-7 (2004).

5. P. Tipler, Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

LECTURE 2

Addition of Vectors

Introduction

We shall demonstrate addition and subtraction of vectors as well as multiplication of a vector by a scalar. We shall also look at some examples of application.

Objective

At the end of this lecture, you should be able to:

- determine the magnitude and direction of added vectors,
- have some working knowledge of multiplication of a vector by a scalar and
- represent a physical vector.

Pre-Test. (See Post-Test)

Definition (Vector Addition)

Vector quantities add according to the parallelogram law, that is, if $\vec{v}$ is a vector represented by the directed line segment $\overrightarrow{OA}$ (see Figure 4 below)

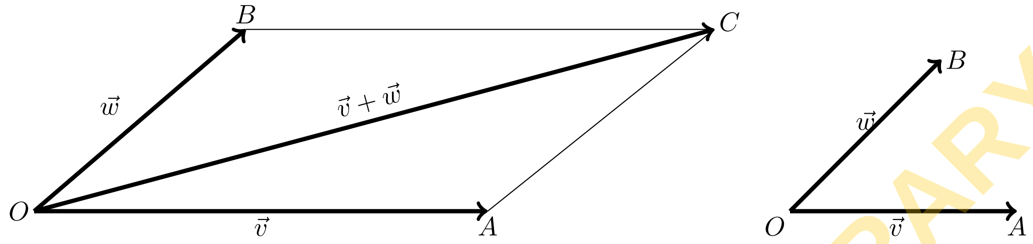


Figure 4:

and $\vec{w}$ is a vector represented by $\overrightarrow{OB}$, then their sum $\vec{v}+\vec{w}$ is defined as the vector represented by $\overrightarrow{OC}$, where $OACB$ is the completed parallelogram (see Figure 4 above).

Example

Two vectors $\vec{u}$ and $\vec{v}$ have a common initial point and form an angle of 60^0. If $|\vec{u}| = 6$ and $|\vec{v}| = 8$, find $|\vec{u}+\vec{v}|$and the angle between $\vec{u}+\vec{v}$ and $\vec{v}$.

Solution(see Figure 5 below)
In $\triangle QRS$,

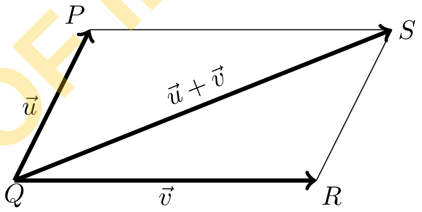


Figure 5:

$$\begin{aligned} QR &= 8,\ RS = 6,\ \angle QRS = 120. \\ QS^2 &= RQ^2 + RS^2 - 2RQ \cdot RS \cos \angle QRS \\ &= 64 + 36 + 2(8)(6)\cos 60^0 \\ &= 148 \\ QS &= 2\sqrt{37} \simeq 12.2 \end{aligned}$$

Hence, $|\vec{u}+\vec{v}| \simeq 12.2$.

$$\begin{aligned}
\frac{\sin\angle SQR}{RS} &= \frac{\sin\angle QRS}{QS}\\
\frac{\sin\angle SQR}{6} &= \frac{\sin 120^0}{12.2}\\
\sin\angle SQR &= \frac{6\sin 60^0}{12.2}\\
&\simeq \frac{6\sqrt{3}}{12.2\times 2}\\
&\simeq .426
\end{aligned}$$

Therefore, $\angle SQR \simeq 25^0$.
The angle between $\vec{u}+\vec{v}$ and $\vec{v}$ is 25^0.

Remark:

When three or more vectors are involved in a problem(see Figure 6 below) they are, in general, not in the same plane. Any two vectors, however, are coplanar, and in certain circumstances, three or more vectors may be coplanar.

For instance, if $\vec{u}, \vec{v}$ and $\vec{w}$ are three coplanar vectors, as in the diagram below,(see Figure 7)

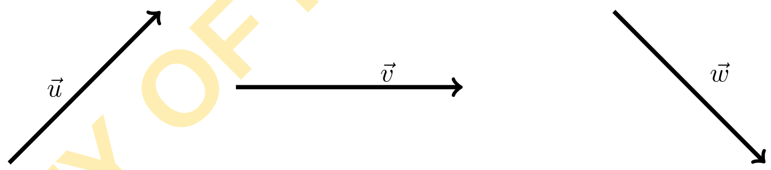


Figure 6:

they may be represented by the equivalent parallel vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$. Then

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$$

and

$$\begin{aligned}
\overrightarrow{PS} &= \overrightarrow{PR} + \overrightarrow{RS}\\
&= \left(\overrightarrow{PQ} + \overrightarrow{QR}\right)\overrightarrow{RS}\\
&= (\vec{u}+\vec{v}) + \vec{w}
\end{aligned}$$

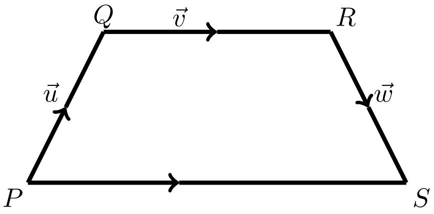


Figure 7:

In the same way, if $\vec{u}, \vec{v}$ and $\vec{w}$ are not coplanar, as in the diagram(see Figure 8 below)

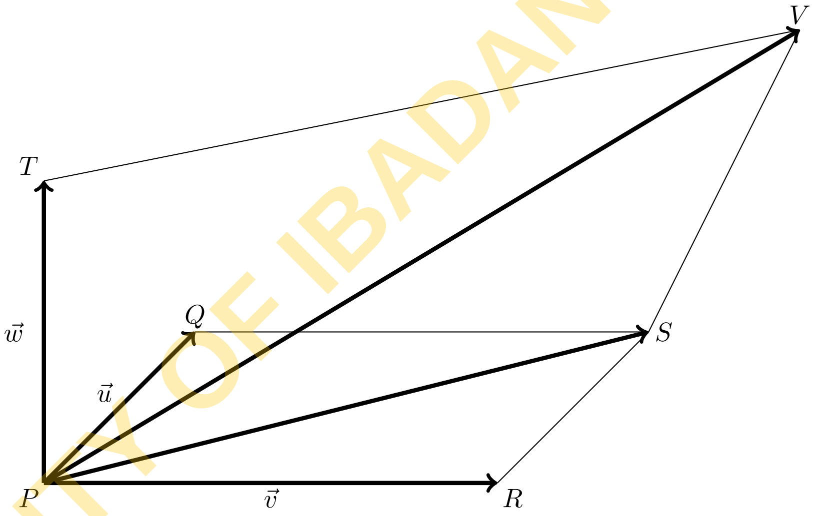


Figure 8:

then

$$\overrightarrow{PV} = (\vec{u} + \vec{v}) + \vec{w}$$

$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are coplanar in the plane $PQSR$ and
$\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{PR}$.
$\overrightarrow{PS}$and $\overrightarrow{PT}$ are coplanar in the plane $PSVT$ and
$\overrightarrow{PV} = \overrightarrow{PS} + \overrightarrow{PT}$.

Therefore,

$$\begin{aligned}\overrightarrow{PV} &= \overrightarrow{PS}+\overrightarrow{PT}\\ &= \left(\overrightarrow{PQ}+\overrightarrow{PR}\right)+\overrightarrow{PT}\\ &= (\vec{u}+\vec{v})+\vec{w}.\end{aligned}$$

Practice Exercise

Two vectors $\vec{u}$ and $\vec{v}$ have a common initial point and form an angle of 75^0. A third vector $\vec{w}$ has the same initial point and is perpendicular to the plane containing $\vec{u}$ and $\vec{v}$. If $|\vec{u}|=4$, $|\vec{v}|=7$ and $|\vec{w}|=6$, calculate $|(\vec{u}+\vec{v})+\vec{w}|$.

It can be deduced immediately from the above definition of vector addition that the following laws hold:

(i) **Commutative Law:** $\vec{u}+\vec{w}=\vec{w}+\vec{v}$, that is, the result of adding two vectors quantities is independent of the order in which the operation is carried out. From this result it follows that, if a point undergoes two successive displacements, then the two displacements may also be regarded as taking place simultaneously.

Note:

(a) Not all quantities possessing both a magnitude and a direction are vector quantities, for not all directed quantities obey the above law of addition, for example, finite rotations of a rigid body free to rotate about a fixed point do not.

(b) The necessary and sufficient condition for a directed quantity to be a vector is that it should combine with another directed quantity of the same kind according to the parallelogram law.

(ii) **Associative Law:** $\vec{v}+(\vec{w}+\vec{u})=(\vec{v}+\vec{w})+\vec{u}$, that is, in the addition of more than two vectors by the repeated use of the above law, the order in which the vectors are associated is irrelevant.
The results are clearly illustrated in Figures 4 and 5.

(iii) **Identity Law:** If Q and R coincide (see Figure 9 below).

Figure 9:

so that

$$\begin{aligned}\vec{v} &= \overrightarrow{QR} = \overrightarrow{QQ}, \text{ then}\\ \vec{u}+\vec{v} &= \overrightarrow{PQ} + \overrightarrow{QQ} = \overrightarrow{PQ}.\end{aligned}$$

Vector $\vec{v}$ or $\overrightarrow{QQ}$ is, in this case, defined to be the zero vector, denoted by $\vec{o}$, and we can write

$$\vec{u} + \vec{o} = \vec{u}$$

Remark:

Although the zero vector has no direction, it is convenient to regard it as parallel to every vector. The zero vector has magnitude zero.

Definition (Vector Subtraction)

To subtract a vector $\vec{v}$ from a vector $\vec{w}$, we add to $\vec{w}$ a vector having the same magnitude as $\vec{v}$, but whose direction is directly opposite to that of $\vec{v}$.
Thus if $\overrightarrow{OQ} = \vec{v}$ and $\overrightarrow{OB} = \vec{w}$ (see Figure 10 below),

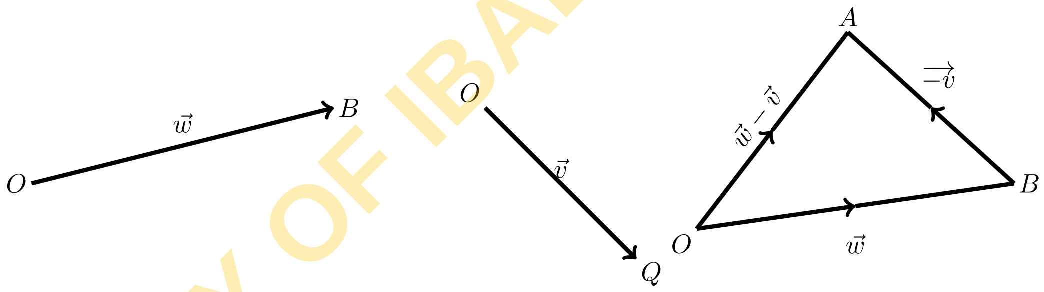


Figure 10:

the vector $\vec{w} - \vec{v}$ is a vector equal and parallel to that represented by the diagonal $\overrightarrow{OA}$.
This construction implies that $(-\vec{v})$ is the vector having the same magnitude as $\vec{v}$, but whose direction is the opposite of that of $\vec{v}$.
The law of vector subtraction is therefore such that the equation

$$\vec{v} - \vec{v} = \vec{o}$$

holds, just as the similar relation holds for scalars.

Example

Prove that

$$(\vec{u}+\vec{v})+(\vec{w}+(-\vec{u}))=\vec{u}+\vec{w}$$

Solution:

$$\begin{aligned}(\vec{u}+\vec{v}+(\vec{w}+(-\vec{u})) &= (\vec{v}+\vec{u})+(-\vec{u}+\vec{w}) \text{ (}\underline{\text{Commutative property}}\text{ used twice)}\\ &= [(\vec{v}+\vec{u})+(-\vec{u})]+\vec{w} \text{ (}\underline{\text{Associative property}}\text{)}\\ &= [\vec{v}+(\vec{u}+(-\vec{u})]+\vec{w} \text{ (Associative property)}\\ &= [\vec{v}+\circ]+\vec{w}\ \text{ (}\underline{\text{inverse property}}\text{)}\\ &= \vec{v}+\vec{w}\ \text{ (}\underline{\text{Identity property}}\text{)}\end{aligned}$$

The result may be indicated diagrammatically.(See Figure 11 below)

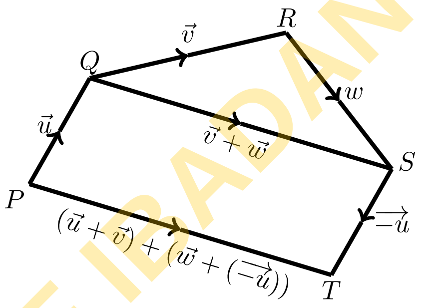


Figure 11:

$$\overrightarrow{QS}=\vec{v}+\vec{w}. \quad \overrightarrow{PT}=(\vec{u}+\vec{v})+(\vec{w}+(-\vec{u}))$$

$PQST$ is a parallelogram; therefore

$$\overrightarrow{QS}=\overrightarrow{PT}.$$

Practice Exercise

Prove that

$$(\vec{u}+\vec{v})+(-\vec{u})=\vec{v}.$$

Definition (Multiplication of a vector by a scalar)

The vector $\vec{v}+\vec{v}$ is clearly a vector having the same direction as the vector $\vec{v}$ but of twice its length. Thus it is denoted by $2\vec{v}$.

Similarly, $m\vec{v}$ is a vector having the same direction as $\vec{v}$ but of length $m\vec{v}$. In particular, if $\vec{v}$ is a unit vector, then $m\vec{v}$ is a vector having the direction of $\vec{v}$ and of the length m.

It immediately follows from this definition that

$$m(n\vec{v}) = n(m\vec{v}) = nm\vec{v}.$$

It can also be shown quite simply, using the properties of similar triangles, that

$$m(\vec{v}+\vec{w}) = m\vec{v} + m\vec{w}.$$

For let ABC be the triangle (see Figure 12 below)

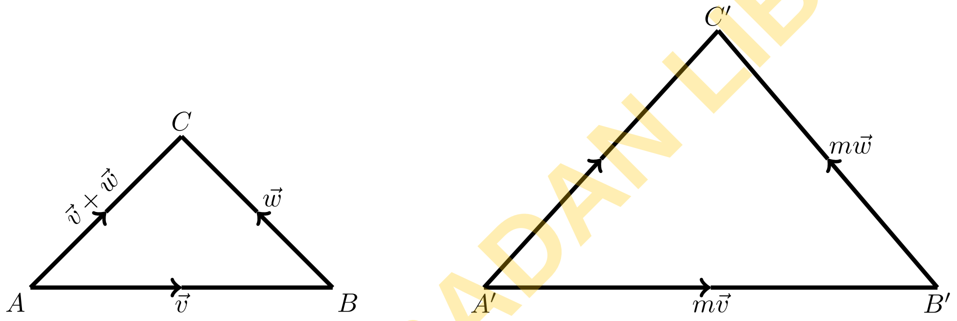


Figure 12:

such that $\overrightarrow{AB} = \vec{v}$, $\overrightarrow{BC} = \vec{w}$.
Then we have, by the law of vector addition,

$$\vec{v}+\vec{w} = \overrightarrow{AC}$$

Now draw the triangle $A'B'C'$ such that

$$\overrightarrow{A'B'} = m\vec{v}, \quad \overrightarrow{B'C'} = m\vec{w}.$$

Then, by the same law, we have

$$m\vec{v} + m\vec{w} = \overrightarrow{A'C'}$$

But since

$$\overrightarrow{A'B'} : \overrightarrow{AB} = \overrightarrow{B'C'} : \overrightarrow{BC}$$

by construction, the triangles are similar, from which it follows that $\overrightarrow{A'C'}$ and $\overrightarrow{AC}$ are parallel (and so have the same direction), while $\overrightarrow{A'C'} = m\overrightarrow{AC}$. Hence,

$$\overrightarrow{A'C'} = m\overrightarrow{AC},$$

i.e.

$$m\vec{v} + m\vec{w} = m(\vec{v} + \vec{w})$$

Example

$ABCD$ is a quadrilateral with $\overrightarrow{AD} = \vec{u}$,
$\overrightarrow{AB} = \vec{v}$, $\overrightarrow{AC} = \vec{u} + 2\vec{v}$.
Express $\overrightarrow{BC}$ and $\overrightarrow{DC}$ in terms of $\vec{u}$ and $\vec{v}$ (see Figure 13 below)

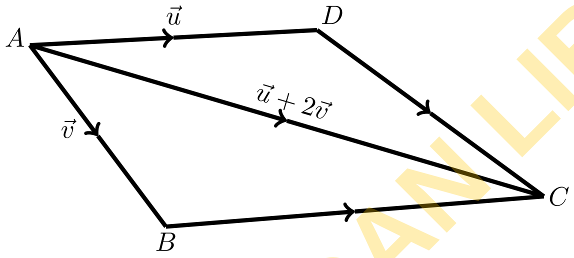


Figure 13:

Solution:

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{AC} - \overrightarrow{AB}\\ &= \vec{u} + 2\vec{v} - \vec{v}\\ &= \vec{u} + \vec{v}.\\ \overrightarrow{DC} &= \overrightarrow{AC} - \overrightarrow{AD}\\ &= \vec{u} + 2\vec{v} - \vec{u}\\ &= 2\vec{v}.\end{aligned}$$

Practice Exercise

$\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ are three vectors which are mutually perpendicular. If $\overrightarrow{OP} = 2\vec{u}$, $\overrightarrow{OQ} = 3\vec{v}$ and $\overrightarrow{OR} = 4\vec{w}$, express $\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}$ in terms of $\vec{u}, \vec{v}$ and $\vec{w}$. If $|\vec{u}| = 2$, $|\vec{v}| = 1$, and $|\vec{w}| = 3$, calculate $|\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}|$.

Linear Combination of Vectors

Definition (Collinear vectors)

Two coplanar vectors are said to be collinear if and only if one is a scalar multiple of the other.

For example, if $\vec{v} = k\vec{u}$, then $\vec{v}$ and $\vec{u}$ are collinear.

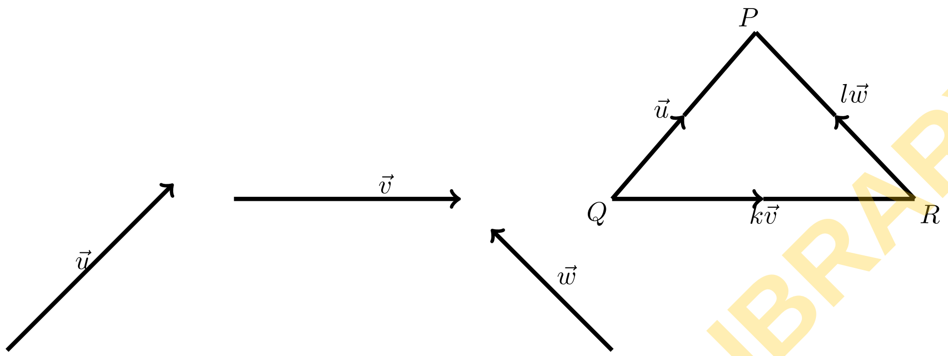


Figure 14:

If we consider three coplanar, noncollinear vectors $\vec{u}, \vec{v}$ and $\vec{w}$ as shown,(see Figure 14(a)) it is possible to construct $\triangle PQR$ (see Figure 14(b))in which $\overrightarrow{QP} = \vec{u}$, $\overrightarrow{QR} = k\vec{v}$, and $\overrightarrow{RP} = l\vec{w}$; that is, $\overrightarrow{QR}$ is parallel to $\vec{v}$ and $\overrightarrow{RP}$ is parallel to $\vec{w}$.

$$\overrightarrow{QP} = \overrightarrow{QR} + \overrightarrow{RP}.$$

Therefore,

$$\vec{u} = k\vec{v} + l\vec{w}.$$

This construction is possible for any three coplanar, noncollinear vectors, and we say that $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$.

Remark:

If $\vec{u}$ and $\vec{v}$ are collinear (or parallel) vectors, and $\vec{w}$ is not parallel to $\vec{u}$ and $\vec{v}$, then $\vec{w}$ is not a linear combination of $\vec{u}$ and $\vec{v}$,(see Figure 15 below)

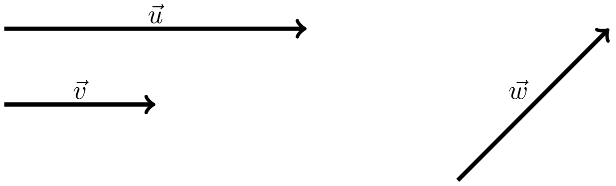


Figure 15:

but $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$, and $\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{w}$.

Definition

Three vectors are coplanar if and only if at least one is a linear combination of the other two.

Theorem:

If $\vec{u} = \overrightarrow{OP}$, $\vec{w} = \overrightarrow{OQ}$, and $\vec{v} = \overrightarrow{OR}$ are three vectors with P, Q and R collinear, and $\overrightarrow{PQ} = k\overrightarrow{QR}$, then (see Figure 16 below)

$$\vec{w} = \frac{k\vec{v} + \vec{u}}{k+1}$$

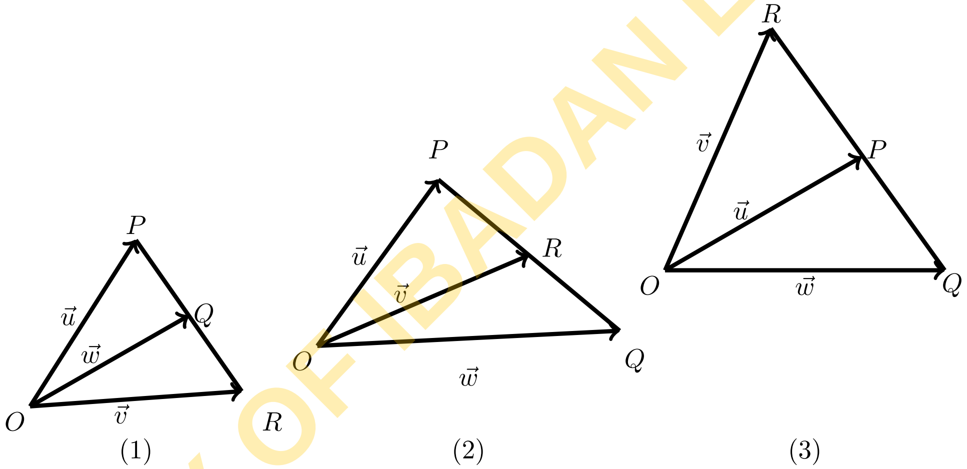


Figure 16:

In case (1), $k > 0$ and Q is between P and R.
In case (2), $|k| > 1$, since $|\overrightarrow{PQ}| > |\overrightarrow{QR}|$, and k is negative because $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are opposite in direction; thus $k < -1$.
In case (3), $|k| < 1$, since $|\overrightarrow{PQ}| < |\overrightarrow{QR}|$, and k is negative because $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are opposite in direction; thus $-1 < k < 0$.

Note that $\overrightarrow{PQ} = k\overrightarrow{QR}$ may be written in the form $\overrightarrow{PQ} : \overrightarrow{QR} = k : 1$, which shows that Q divides the line segment PR in the ratio $k : 1$.

Proof:

P, Q, R are collinear and

$$\overrightarrow{PQ} = k\overrightarrow{QR}.$$

Therefore,

$$\overrightarrow{PO}+\overrightarrow{OQ}=k\left(\overrightarrow{QO}+\overrightarrow{OR}\right).$$

Hence,

$$\begin{aligned}-\vec{u}+\vec{w} &= k(-\vec{w}+\vec{v})\\ \vec{w}+k\vec{w} &= k\vec{v}+\vec{u}\end{aligned}$$

$$\vec{w}=\frac{k\vec{v}+\vec{u}}{k+1} \qquad \square$$

Note that if Q is the midpoint of PR so that $k=1$, then

$$\vec{w}=\frac{\vec{v}+\vec{u}}{2}=\frac{1}{2}\vec{v}+\frac{1}{2}\vec{u}.$$

Since all the steps in the solution are reversible, the converse theorem may be proved by simply reversing the steps in the solution.

Converse Theorem

If $\vec{u}=\overrightarrow{OP}$, $\vec{w}=\overrightarrow{OQ}$, and $\vec{v}=\overrightarrow{OR}$ are such that

$$\vec{w}=\frac{k\vec{v}+\vec{u}}{k+1},$$

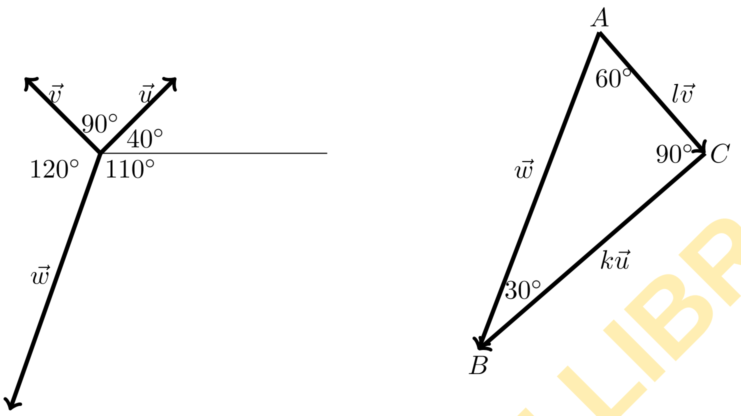
then P, Q and R are collinear, and Q divides the line segment PR in the ratio $k:1$.

Practice Exercise

Prove the above converse Theorem.

Example

Three coplanar vectors $\vec{u},\vec{v}$ and $\vec{w}$ make angles of 40^0, 130^0, and 250^0, respectively, with the horizontal. If $|\vec{u}|=4$, $|\vec{v}|=6$ and $|\vec{w}|=12$, express $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$.
(See Figure $17(a)$ and Figure $17(b)$, respectively)



(a) (b)

Figure 17:

Solution:

Let

$$\vec{w} = l\vec{v} + k\vec{u}$$

In $\triangle ABC$, $AB = 12$, $AC = 6|l|$, $BC = |k|$,
$\angle ABC = 30^0$, $\angle ACB = 90^0$, $\angle BAC = 60^0$

$$AC = AB \sin 30^0.$$

Thus,

$$\begin{aligned} 6|l| &= 12 \times \frac{1}{2} \\ |l| &= 1. \\ BC &= AB \cos 30^0 \\ 4|k| &= \frac{12\sqrt{3}}{2} \end{aligned}$$

Therefore,

$$|k| = \frac{3\sqrt{3}}{2}.$$

From the diagram, $k < 0$, $l < 0$; therefore $l = -1$, $k = -\frac{3\sqrt{3}}{2}$, and $\vec{w} = -\vec{v} - \frac{3\sqrt{3}}{2}\vec{u}$.
Note also that

$$\frac{3\sqrt{3}}{2}\vec{u} = -\vec{v} - \vec{w}.$$

Therefore,

$$\begin{aligned}\vec{u} &= -\frac{2}{3\sqrt{3}}\vec{v} - \frac{2}{\sqrt{3}}\vec{w} \\ &= -\frac{2\sqrt{3}}{9}(\vec{v}+\vec{w}).\end{aligned}$$

Summary
We have covered the following in this lecture.

Definitions
If $\vec{u}$ and $\vec{v}$ are two vectors represented by $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, respectively, so that the endpoint of the line segment PQ is the initial point of the line segment QR, then the sum of $\vec{u}$ and $\vec{v}$ is represented by $\overrightarrow{PR}$.
The negative of a vector $\vec{u}$ is a vector equal in magnitude to u but opposite in direction, and is denoted by $-\vec{u}$.
If $\vec{u}$ is a nonzero vector and k a nonzero real number (scalar), the vector $k\vec{u}$ is defined by the following rules
(1) $|k\vec{u}| = |k||\vec{u}|$
(2) The directions of $k\vec{u}$ and $\vec{u}$ are the same if $k > 0$ and opposite if $k < 0$. Also $k\circ = \circ$ and $0\vec{u} = \circ$.
Two coplanar vectors are collinear if one is a scalar multiple of the other.
Three vectors are coplanar if at least one is a linear combination of the other two.
Algebraic properties of Vectors.
For the following properties, $\vec{u}, \vec{v}, \vec{w}$ are vectors.
(1) $\vec{u}+\vec{v}$ is a vector Closure property
(2) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ Commutative property.
(3) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{u}+\vec{w})$. Associative property.
(4) $\vec{u}+\circ = \circ+\vec{u} = \vec{u}$ Identity property (zero vector)
(5) $\vec{u}+(-\vec{u}) = -\vec{u}+\vec{u} = \circ$ Inverse property.
For $\vec{u}$ and $\vec{v}$ are vectors and k and l are real numbers.
(6) $k(l\vec{v}) = l(k\vec{v}) = lk\vec{v}$
(7) $k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}$.
Theorem
If $\vec{u} = \overrightarrow{OP}$, $\vec{w} = \overrightarrow{OQ}$, and $\vec{v} = \overrightarrow{OR}$ are three vectors with P, Q and R collinear and $\overrightarrow{PQ} = k\overrightarrow{QR}$, then
$w = \dfrac{k\vec{v}+\vec{u}}{k+1}$

Post-Test.

1. Two vectors $\vec{u}$ and $\vec{v}$ have a common initial point and form an angle of 140^0. If

$|\vec{u}| = 4$ and $|\vec{v}| = 8$, calculate, to the nearest integer, the value of $|\vec{u} + \vec{v}|$ and, to the nearest degree, the size of the angle between $\vec{u} + \vec{v}$ and $\vec{v}$.

2. Simplify $3\vec{u} + 2\vec{v} - 2(\vec{v} - \vec{u}) + (-3\vec{u})$.
3. Three coplanar vectors $\vec{u}, \vec{v}$ and $\vec{w}$ make angles of 30^0, 60^0 and 120^0, respectively, with the horizontal. If $|\vec{u}| = 5$, $|\vec{v}| = 6$, $|\vec{w}| = 8$, express $\vec{v}$ as a linear combination of $\vec{u}$ and $\vec{w}$.
4. If $\overrightarrow{OQ} = \frac{3}{7}\overrightarrow{OP} + \frac{4}{7}\overrightarrow{OR}$, prove that P, Q, R are collinear and that $\overrightarrow{PQ} : \overrightarrow{QR} = 4 : 3$.
5. If $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are such that A, B and C are collinear and $\overrightarrow{AB} : \overrightarrow{BC} = -5 : 3$, express $\overrightarrow{OB}$ as a linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.
6. If $\overrightarrow{OP} = 3\overrightarrow{OQ} - 2\overrightarrow{OR}$, prove that P, Q and R are collinear, and find the value of the ratio $\overrightarrow{PQ} : \overrightarrow{QR}$.
7. $PQRS$ is a quadrilateral with $\overrightarrow{PQ} = 2\vec{u}$, $\overrightarrow{QR} = 3\vec{v}$, $\overrightarrow{QS} = 3\vec{v} - 3\vec{u}$. Express $\overrightarrow{PS}$ and $\overrightarrow{RS}$ in terms of $\vec{u}$ and $\vec{v}$.

Supplementary Reading

1. M.R. Spiegel, Theory and Problems of Vector Analysis, Schaum Publishing Co., New York (1959).

2. P. Tipler, Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

LECTURE 3

The Scalar Product of Two Vectors

Introduction

This is defined as the product of the absolute value of either into the orthogonal projection of the other on it.

It clearly has not direction properties and so is a scalar quantity.

Objective

At the end of this lecture, you should be able to:

- define the component of one vector in the direction of another vector,
- use dot product formular to calculate the component of one vector in the direction of another vector,
- Calculate dot product of the vectors from their respective components,
- Write a vector as the sum of another vector parallel to a fixed vector and another vector perpendicular to fixed vector,
- demonstrate how a given vector is perpendicular to the given line in a given planned, and
- determine the length of the vector projection of one vector onto another.

Pre-Test. (See Post-Test)

Definition (Scalar Product or Inner Product)

The scalar product of two vectors $\vec{v}$ and $\vec{w}$ is a scalar defined by the equation

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta, \tag{1}$$

where θ measures the smallest angle determined by $\vec{v}$ and $\vec{w}$ when their initial points coincide, as in Figure 18 below

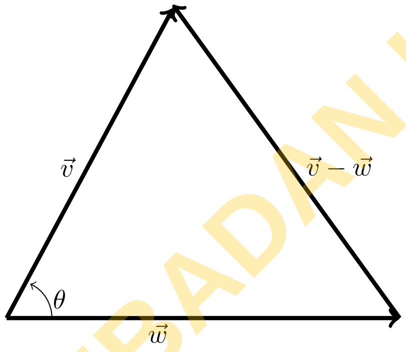


Figure 18:

This product is also called the dot product, because of the dot used to denote it.

Remark

(i) The definition of the dot product given here is "coordinate-free", in the sense that it is independent of whatever reference frame we might use to describe the vectors in terms of coordinates.

(ii) The operation of scalar multiplication is commutative. That is,

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

Definition (Vector projection)

The vector we get by projecting $\vec{w}$ onto the line through $\vec{v}$ is called the vector projection of $\vec{w}$ onto $\vec{v}$. We denote it by $\text{Proj}_{\vec{v}\vec{w}}$ (see Figure 18 above).

Definition (Component of vector)

The component of $\vec{w}$ in the direction of $\vec{v}$ is a number that is plus or minus the length of the vector projection of $\vec{w}$ onto $\vec{v}$. The sign is plus if $\text{Proj}_{\vec{v}\vec{w}}$ has the same direction as $+\vec{v}$, and is minus if it has the same direction as $-\vec{v}$.

In either case, the component of $\vec{w}$ in the direction of $\vec{v}$ is equal to $|\vec{w}|\cos\theta$ (See Figure 18 again).

Remark

The dot product gives a convenient way to calculate the component of $\vec{w}$ in the direction of $\vec{v}$.

We solve Eqn (1) for $|\vec{w}|\cos\theta$ to get

$$\vec{w} - \text{component in } \vec{v}\text{-direction} = |\vec{w}|\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|} \tag{3}$$

To calculate $\vec{v}\cdot\vec{w}$ form the components of $\vec{v}$ and $\vec{w}$, we let

$$\begin{aligned} \vec{v} &= v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \\ \vec{w} &= w_1\vec{i} + w_2\vec{j} + w_3\vec{k} \end{aligned} \tag{4}$$

and

$$\begin{aligned} \vec{u} &= \vec{w} - \vec{v} \\ &= (w_1 - v_1)\vec{i} + (w_2 - v_2)\vec{j} + (w_3 - v_3)\vec{k} \end{aligned}$$

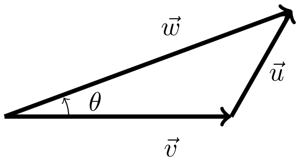


Figure 19:

Then we apply the law of cosines to a triangle whose sides represent the vectors $\vec{v}, \vec{w}$ and $\vec{u}$ (see Figure 19 above) and obtain

$$|\vec{u}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta,$$

$$|\vec{v}||\vec{w}|\cos\theta = \frac{|\vec{v}|^2 + |\vec{w}|^2 - |\vec{u}|^2}{2} \tag{5}$$

The result of this algebra is the formula

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \tag{6}$$

Thus, to find the scalar product of two given vectors we multiply their corresponding components together and add the results.

Example: Find the angle θ between $\vec{v} = \vec{i} - 2\vec{j} - 2\vec{k}$ and $\vec{w} = 6\vec{i} + 3\vec{j} + 2\vec{k}$. Also, find the component of $\vec{w}$ in the direction of $\vec{v}$.

Solution

$$\vec{v} \cdot \vec{w} = 6 - 6 - 4 = -4$$

From eqn (1), since $|\vec{v}| = \sqrt{1+4+4} = 3$ and $|\vec{w}| = \sqrt{36+9+4} = 7$, we have

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{-4}{21},$$

$$\theta = \cos^{-1}\left(\frac{-4}{21}\right) \simeq 101^0$$

The component of $\vec{w}$ in the direction of $\vec{v}$ is

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{v}|} = -\frac{4}{3}.$$

This is the negative of the length of the vector projection of $\vec{v}$ onto $\vec{w}$.

Practice Exercise

Find the component of $\vec{v} = 2\vec{i} + 2\vec{j} + \vec{k}$ in the direction of $\vec{w} = 2\vec{i} + 10\vec{j} - 11\vec{k}$.

Scalar multiplication obeys the distributive laws:

(i) $\vec{v} \cdot) \vec{w} + \vec{u}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{u}$

(ii) $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$

(iii) $(\vec{v} + \vec{w}) \cdot (\vec{u} + \vec{p}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{p} + \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{p}$

Practice Exercise

Show that (i), (ii) and (iii) are true.

Special cases

(i) When $\theta = 0^0$, that is, when the two vectors are coincident or parallel,

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|;$$

(ii) When $\theta = 90^0$, that is, when the two vectors are orthogonal (or perpendicular)

$$\vec{v} \cdot \vec{w} = 0$$

Hence, if $(\vec{i}, \vec{j}, \vec{k})$ are three unit vectors defining an orthogonal system of reference, then we have

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1;$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = 0, \;\; \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{j} = 0;$$

$$\vec{k} \cdot \vec{i} = \vec{i} \cdot \vec{k} = 0.$$

Example: Write the vector $\vec{w}$ as the sum of a vector $\vec{w_1}$ parallel to $\vec{v}$ and a vector $\vec{w_2}$ perpendicular to $\vec{v}$ (see Figure 20 below).

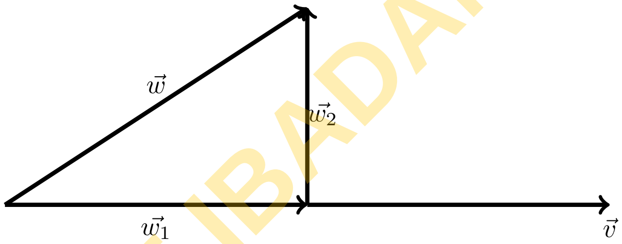


Figure 20:

Solution:

Let

$$\vec{w} = \vec{w_1} + \vec{w_2},$$

with $\vec{w_1} = c\vec{v}$ and $\vec{w_2} \cdot \vec{v} = 0$. Then, substituting $c\vec{v}$ for $\vec{w_1}$, we have

$$\vec{w} = c\vec{v} + \vec{w_2}$$

and

$$0 = \vec{w_2} \cdot \vec{v} = (\vec{w} - c\vec{v}) \cdot \vec{v} = \vec{w} \cdot \vec{v} - c(\vec{v} \cdot \vec{v})$$

or

$$c = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

Then,

$$\vec{w_2} = \vec{w} - \vec{w_1} = \vec{w} - c\vec{v} = \vec{w} - \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}$$

is perpendicular to $\vec{v}$ because c was chosen to make $\vec{w_2} \cdot \vec{v} = 0$.
For example, if

$$\vec{w} = 2\vec{i} + \vec{j} - 3\vec{k} \text{ and } \vec{v} = 3\vec{i} - \vec{j},$$

then

$$c = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{6-1}{9+1} = \frac{1}{2}$$

and

$$w_1 = \frac{1}{2}\vec{v} = \frac{3}{2}\vec{i} - \frac{1}{2}\vec{j}$$

is parallel to $\vec{v}$, while

$$\vec{w_2} = \vec{w} - \vec{w_1} = \frac{1}{2}\vec{i} + \frac{3}{2}\vec{j} - 3\vec{k}$$

is perpendicular to $\vec{v}$.

Example: Show that the vector $\vec{n} = a\vec{i} + b\vec{j}$ is perpendicular to the line $ax + by = c$ in the xy-plane (see Figure 21 below).

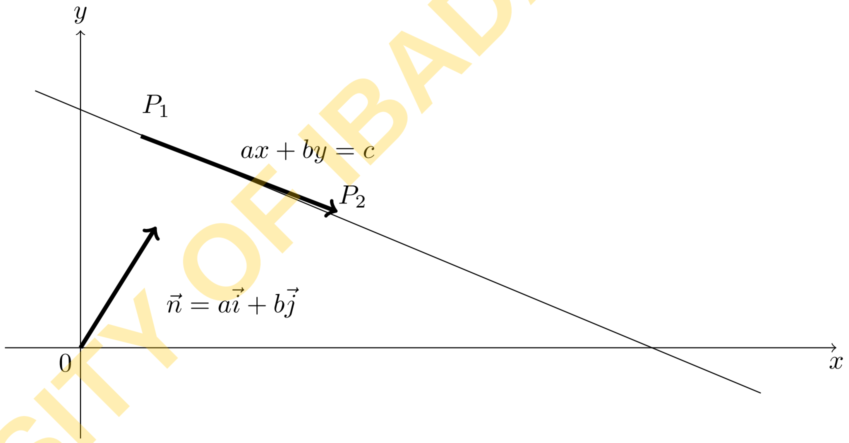


Figure 21:

Solution:

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points on the line; that is,

$$ax_1 + by_1 = c, \quad ax_2 + by_2 = c.$$

By subtraction, we eliminate c and obtain

$$a(x_2 - x_1) + b(y_2 - y_1) = 0.$$

or

$$(a\vec{i}+b\vec{j})\cdot[(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}]=0 \tag{7}$$

Now, $(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}=\overrightarrow{p_1p_2}$ is a <u>vector</u> joining two points on the line, while $\vec{n}=a\vec{i}+b\vec{j}$ is the given vector. Equation (7) says that either $\vec{n}=0$ or $\overrightarrow{p_1\cdot p_2}=0$, or else $\vec{n}\perp\overrightarrow{p_1p_2}$. But $ax+by=c$ is assumed to be an honest equation of a straight line, so that a and b are not both zero and $\vec{n}\neq 0$.

Furthermore, we may surely choose p_2 different from p_1 on the line to make $\overrightarrow{p_1p_2}\neq 0$. Hence, $\vec{n}\perp\overrightarrow{p_1p_2}$.
For example, $\vec{n}=2\vec{i}-3\vec{j}$ is <u>normal</u> to the line $2x-3y=5$.

Practice Exercise

Using vector methods, find the <u>distance</u> d between the point $P(4,3)$ and the line $L: x+3y=6$ (see Figure 22 below)

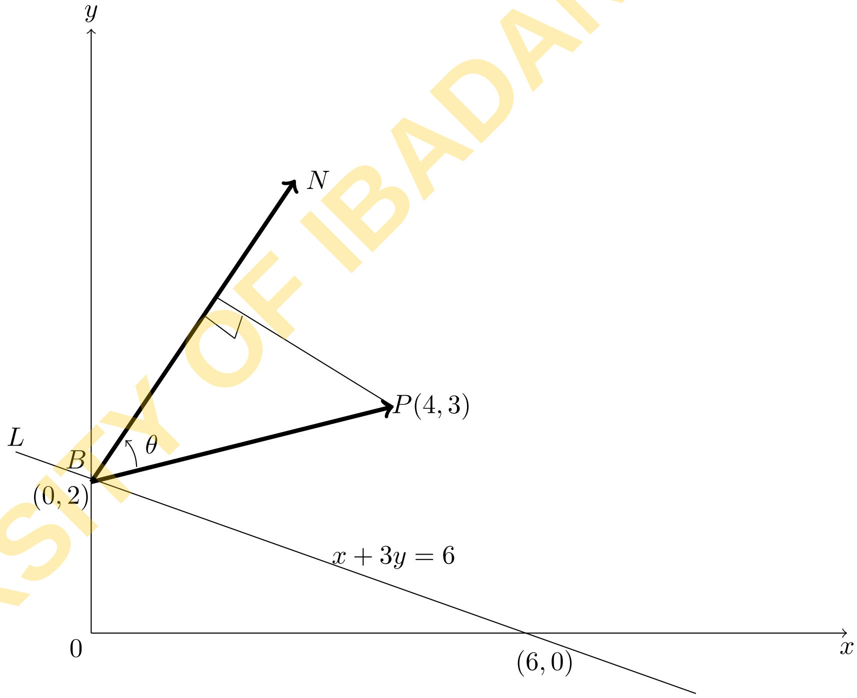


Figure 22:

Application

The dot product is useful in mechanics, where it is used in calculating the work done by a force F when the point of application of F undergoes a displacement $\overrightarrow{AB}$. If the force remains constant in direction and magnitude, this work is given by (see Figure 23 below).

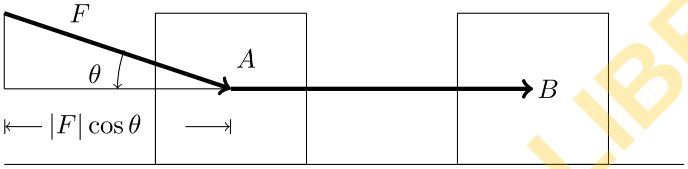


Figure 23:

$$\begin{aligned}\text{Work} &= (|\vec{F}|\cos\theta)|\overrightarrow{AB}| \\ &= \vec{F}\cdot\overrightarrow{AB}.\end{aligned}$$

Remark

The concept of work also enters into the study of electricity and magnetism and the scalar product again plays a basic role.

Summary

We have covered the following in this lecture.

Definitions

The scalar product of two vectors $\vec{v}$ and $\vec{w}$ is a scalar defined by the equation

$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$

The vector projection of $\vec{w}$ onto $\vec{v}$ is the vector that we get by projecting $\vec{w}$ onto the line through $\vec{v}$

The component of $\vec{w}$ in the direction of $\vec{v}$ is a number that is plus or minus the length of the vector projection of $\vec{w}$ onto $\vec{v}$.

Algebraic properties of scalar product.

(1) $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$ (Commutative property)

(2) $\vec{w}$-component in $\vec{v}$-direction $= |\vec{w}|\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}$

(3) If $\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{w} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$
then $\vec{v}\cdot\vec{w} = a_1b_1 + a_2b_2 + a_3b_3$.

(4) Distributive laws hold
(i) $\vec{v}\cdot(\vec{w}+\vec{u}) = \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{u}$
(ii) $(\vec{v}+\vec{w})\cdot\vec{u} = \vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u}$
(iii) $(\vec{v}+\vec{w})\cdot(\vec{u}+\vec{p}) = \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{p} + \vec{w}\cdot\vec{u} + \vec{w}\cdot\vec{p}$.

5(i) $\vec{v}\cdot\vec{w} = 0$ if and only if the vectors $\vec{v}$ and $\vec{w}$ are perpendicular

(ii) If $\vec{v}\cdot\vec{w} < 0$, then $\cos\theta$ is negative and the angle between the vectors is greater than 90^0.

(iii) If $\vec{v} = \vec{w}$, then $\theta = 0$ and $\cos\theta = 1$, so that $\vec{v}\cdot\vec{v} = |\vec{v}|^2$.

Calculation:

Write the vector $\vec{w}$ as the sum of vectors parallel and perpendicular to $\vec{v}$.

Explain how the vector $\vec{n} = a\vec{i} + b\vec{j}$ is normal to the line $ax + by = c$

Find the distance between a point and a line on the xy-plane

Find the work done by the formula

Work $= (|\vec{F}|\cos\theta)|\overrightarrow{AB}| = \overline{F}\cdot\overrightarrow{AB}$

where $\vec{F}$ is force, $\overrightarrow{AB}$ is a displacement at the application $\vec{F}$.

Post-Test

1. Suppose it is known that $\vec{v}\cdot\vec{w_1} = \vec{v}\cdot\vec{w_2}$, and $\vec{v}$ is not zero, but nothing more is known about the vector $\vec{w_1}$ and $\vec{w_2}$. Is it permissible to cancel $\vec{v}$ from both sides of the equation? Give a reason for your answer.

2(a) Express the vector projection of $\vec{w}$ onto $\vec{v}$ in a vector form that is convenient for calculation.

(b) Find the vector projection of $\vec{w} = \vec{i} + 3\vec{j} + 4\vec{k}$ onto the vector $\vec{v} = 10\vec{i} + 11\vec{j} - 2\vec{k}$.

3. Find the interior angles of the triangle ABC whose vertices are the points $A(-1, 0, 2)$, $B(2, 1, -1)$ and $C(1, -2, 2)$.

4. Find the point $A(a, a, 0)$ on the line $y = x$ in the xy-plane such that the vector $\overrightarrow{AB}$ is perpendicular to the line OA. Here O is the origin and B is the point $(2, 4, -3)$.

5. Find the angle between the diagonal of a cube and its edges.

6. If $a = |\vec{A}|$ and $b = |\vec{B}|$, show that the vector

$$\vec{C} = \frac{a\vec{B} + b\vec{A}}{a + b}$$

bisects the angle between $\vec{A}$ and $\vec{B}$.

7. With the same notation as in Problem 6, show that the vectors $a\vec{B} + b\vec{A}$ and $\vec{A}b - \vec{B}a$ are perpendicular.

8. Find the work done by a force $\vec{F} = -\vec{w}k$ as its point of application moves from the point $P_1(x_1, y_1, z_1)$ to a second point $P_2(x_2, y_2, z_2)$ along the straight line P_1P_2.

9. Using vector methods, show that the distance d between the point (x_1, y_1) and the line $ax + by + c = 0$ is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

10. Show that if r is a scalar, then $(r\vec{v})\cdot\vec{w} = r(\vec{v}\cdot\vec{w})$.

Supplementary Reading

1. M.R. Spiegel, Theory and Problems of Vector Analysis, Schaum Publishing Co., New York (1959).

2. P. Tipler, Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

LECTURE 4

The Vector Product of Vectors in Space

Introduction

The vector product (also called the cross product or outer product) is only meaningful in three dimensions. The vector product differs from the scalar product primarily in that the result of the vector product of two vectors is a vector.

In this lecture, we shall define and give properties of the following: Concepts; Vector product of two vectors; Scalar triple product and vector triple product.

Objective
At the end of this lecture you should be able to:
- verify that vector product is a vector,
- verify that vector product multiplication is not commutative.

Reversing the order of the factors changes the product,
- express vector product of two vectors in terms of their components.
- establish the distributive law for vector product
- compute vector product and vector triple product.

Pre-Test (See Post-Test)

Definition and Notations (Vector Product)

The vector product, denoted $\vec{a} \times \vec{b}$, is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ and is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin(\theta)\vec{n}$$

where θ is the measure of the angle between $\vec{a}$ and $\vec{b}$, and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ which completes a right-handed system (see Figure 24 below).

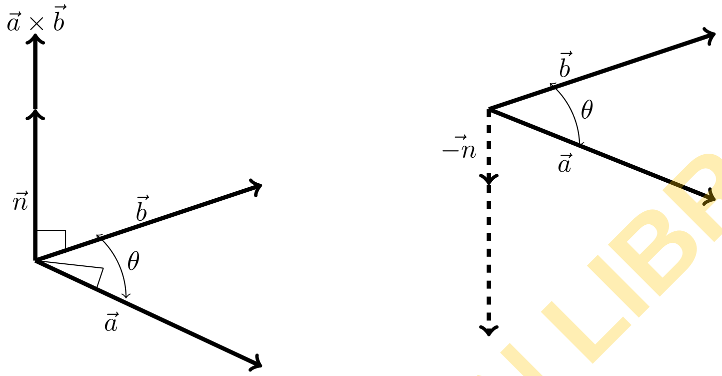


Figure 24:

The right-handedness constraint is necessary because there exist two unit vectors, $\vec{n}$ and $(-\vec{n})$ that are perpendicular to both $\vec{a}$ and $\vec{b}$.
The length of $\vec{a} \times \vec{b}$ can be interpreted as the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as sides.

Remark

From the definition it follows that

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

so that the vector product does not obey the commutative law.

Special Cases

(i) When $\theta = 0^0$, so that $\vec{a}$ and $\vec{b}$ are coincident or parallel, then

$$\vec{a} \times \vec{b} = 0.$$

(ii) When $\theta = 90^0$, and the vectors are orthogonal, then

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\vec{n}$$

where $\vec{n}$ is a unit vector at right angles to both $\vec{a}$ and $\vec{b}$ and whose direction is that of the vector product $\vec{a} \times \vec{b}$.

If $\{\vec{i},\vec{j},\vec{k}\}$ form a set of unit vectors, mutually orthogonal, we therefore have

$$\begin{aligned}\vec{i}\times\vec{i} &= \vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0;\\ \vec{i}\times\vec{j} &= -(\vec{j}\times\vec{i})=\vec{k},\;\; \vec{j}\times\vec{k}=-(\vec{k}\times\vec{j})=\vec{i},\\ \vec{k}\times\vec{i} &= -(\vec{i}\times\vec{k})=\vec{j}.\end{aligned}$$

Remark

The last three results hold, of course, only if $\{\vec{i},\vec{j},\vec{k}\}$ define a right-handed rule, that is, the relative positions of the axes are such that a clockwise rotation about the axis "i" through a right angle brings the axis "j" into the original position of the axis "k", and similarly for rotations about the other axes.

Other algebraic properties, namely:
The vector product is distributive over addition,

$$\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}),$$

and compatible with scalar multiplication so that

$$(r\vec{a})\times\vec{b}=\vec{a}\times(r\vec{b})=r(\vec{a}\times\vec{b}).$$

It is not associative, but satisfies the Jacobi identity:

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0.$$

It does not obey the cancellation law:
If $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ and $\vec{a}\neq 0$ then:
$(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})=0$ and, by the distributive law above:

$$\vec{a}\times(\vec{b}-\vec{c})=0$$

Now, if $\vec{a}$ is parallel to $(\vec{b}-\vec{c})$, then even if $\vec{a}\neq 0$, it is possible that $(\vec{b}-\vec{c})\neq 0$ and therefore that $\vec{b}\neq\vec{c}$.
However, if both $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, then it can be concluded that $\vec{b}=\vec{c}$. Indeed,

$$\vec{a}\cdot(\vec{b}-\vec{c})=0,\;\;\text{and}$$

$$\vec{a}\times(\vec{b}-\vec{c})=0$$

So that $\vec{b}-\vec{c}$ is both parallel and perpendicular to the non-zero vector $\vec{a}$. This is only possible if $\vec{b}-\vec{c}=0$.

The following expression can be used to calculate $\vec{a} \times \vec{b}$ if the components of $\vec{a}$ and $\vec{b}$ are provided:
Suppose

$$\begin{aligned}\vec{a} &= a_1\vec{i} + a_2\vec{j} + a_3\vec{k},\\ \vec{b} &= b_1\vec{i} + b_2\vec{j} + b_3\vec{k},\end{aligned}$$

we obtain

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\\ &= \vec{i}(a_2b_3 - a_3b_2) + \vec{j}(a_3b_1 - a_1b_3) + \vec{k}(a_1b_2 - a_2b_1)\end{aligned}$$

Practice Exercise

Prove that the above expression is true.

Example 1

Find the area of the triangle whose vertices are $A(1, -1, 0)$, $B(2, 1, -1)$, and $C(-1, 1, 2)$ (see Figure 25 below)

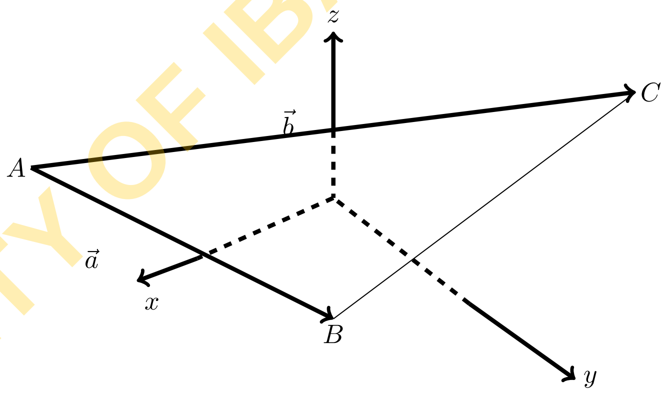


Figure 25:

Solution

Two sides of the given triangle are represented by the vectors

$$\begin{aligned}\vec{a} &= \overrightarrow{AB} = (2-1)\vec{i} + (1+1)\vec{j} + (-1-0)\vec{k} = \vec{i} + 2\vec{j} - \vec{k},\\ \vec{b} &= \overrightarrow{AC} = (-1-1)\vec{i} + (1+1)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{j} + 2\vec{k}.\end{aligned}$$

The area of the triangle is one-half the area of the parallelogram represented by these vectors.
The area of the parallelogram is the magnitude of the vector.

$$\begin{aligned}\vec{c} = \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & 2 & 2 \end{vmatrix} \\ &= \vec{i}\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} = 6\vec{i} + 6\vec{k},\end{aligned}$$

which is $|\vec{c}| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$.
Therefore, the area of the triangle is

$$\frac{1}{2}|\vec{a} \times \vec{b}| = 3\sqrt{2}.$$

Example 2

Find aunit vector perpendicular to both $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + 2\vec{k}$.

Solution: The vector $\vec{n} = \vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. We divide $\vec{n}$ by $|\vec{n}|$ to obtain a unit vector $\vec{u}$ that has the same direction as $\vec{n}$:

$$\begin{aligned}\vec{u} = \frac{\vec{n}}{|\vec{n}|} &= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{\vec{i} - 5\vec{j} - 3\vec{k}}{\sqrt{1^2 + (-5)^2 + (-3)^2}} \\ &= \frac{\vec{i} - 5\vec{j} - 3\vec{k}}{\sqrt{35}}\end{aligned}$$

Either $\vec{u}$ or its negative will do.

Practise Exercise

Find $\vec{a} \times \vec{b}$ if $\vec{a} = 2\vec{i} - 2\vec{j} - \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$.

Definition (Vector triple product or triple product expansion or Lagrange's formula). The triple product expansion, also known as Lagrange's formula, is a formula relating the vector product of three vectors (called vector triple product) with the dot product:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).$$

Which is easier to remember as "*BAC* minus *CAB*", keeping in mind which vectors are dotted together. This formula is commonly used to simplify vector calculations in physics.

Practise Exercise:

Verify above equation for $\vec{a}=\vec{i}-\vec{j}+2\vec{k}$, $\vec{b}=2\vec{i}+\vec{j}+\vec{k}$, $\vec{c}=\vec{i}+2\vec{j}-\vec{k}$. The following identity also relates the vector product and the dot product:

$$|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2.$$

Summary

We have covered the following in this lecture:

1. The vector product, of $\vec{a}$ and $\vec{b}$ is defined by the equation
   $\vec{a}\times\vec{b}=\vec{n}|\vec{a}||\vec{b}|\sin\theta$
   where $\vec{n}$ is a unit vector in $\vec{u}$ direction of $\vec{a}\times\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.
2. The algebraic properties of vector product such as:
   (i) Vector product is not commutative, that is, $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$ so we deduce that from set of unit vectors $\{\vec{i},\vec{j},\vec{k}\}$, we have
   $\vec{i}\times\vec{j}=-(\vec{j}\times\vec{i})=\vec{k}$,
   $\vec{j}\times\vec{k}=-(\vec{k}\times\vec{j})=\vec{i}$,
   $\vec{k}\times\vec{i}=-(\vec{i}\times\vec{k})=\vec{j}$,
   while $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$.
   (ii) The associative law satisfied by vector product is
   $(r\vec{a})\times(s\vec{b})=(rs)\vec{a}\times\vec{b}$.
   (iii) The distributive law satisfied by vector product are:
   $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
   $(\vec{b}+\vec{c})\times\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$
3. The length of $\vec{a}\times\vec{b}$ is given as the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as sides.
4. Vector triple product is defined as:
   $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$
5. An equation relating vector product and dot product is given as:
   $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2$.

Post-Test

1. Find a vector n perpendicular to the plane determined by the points $A(1,-1,2)$, $B(2,0,-1)$, and $C(0,2,1)$.
2. Find the area of the triangle ABC of problem 1.
3. Find the distance between the origin and the plane ABC of Problem 1 by projecting $\overrightarrow{OA}$ onto the normal vector n.
4. Find a vector that is perpendicular to both of the vectors $\vec{a}=\vec{i}+\vec{j}+\vec{k}$ and $\vec{b}=\vec{i}+\vec{j}$.

5. Using vector methods, find the distance between the line L_1 determined by the two points $A(1, 0, -1)$, $B(-1, 1, 0)$ and the line L_2 determined by the points $C(3, 1, -1)$, $D(4, 5, -2)$. The distance is to be measured along a line perpendicular to both L_1 and L_2.

6. $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}$ is normal to a plane M_1 and $\vec{b} = 2\vec{i} - \vec{j} + \vec{k}$ is normal to a second plane M_2.

   (a) Find the angle between the two normals.

   (b) Do the two planes intersect? Give a reason for your answer.

   (c) If the two planes do intersect, find a vector parallel to their line of intersection.

7. Let $\vec{a}$ be a nonzero vector. Show that

   (a) $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does not guarantee $\vec{b} = \vec{c}$

   (b) $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ together imply $\vec{b} = \vec{c}$.

8. Find the volume of the tetrahedron with vertices at $(0, 0, 0), (1, -1, 1), (2, 1, -2)$ and $(-1, 2, -1)$.

Supplementary Reading

1. M.R. Spiegel, Theory and Problems of Vector Analysis, Schaum Publishing Co., New York (1959).

2. F.P. Beer and E.R. Johnston, Vector Mechanics for Engineers (Statics). McGraw-Hill, 3rd Edition (1999).

3. H. David, R. Resnick and J. Walker, Fundamentals of Physics, Wiley; 7 sub edition (June 16, 2004), ISBN 0471232319.

4. W. Kahan, Cross-Products and Rotations in Euclidean 2- and 3-space. University of California, Berkeley (PDF).
(http://www.cs.berkeley.edu/∼wkahan/mathH110/cross.pdf)

LECTURE 5

Elements of Vector Calculus

Introduction

As in ordinary analysis, so in vector theory, the process of differentiating a vector with respect to a scalar as well as integrating vector function shall be extensively discussed in this lecture.

Objective
At the end of this lecture you should be able to:
•define the derivative of a vector with respect to a scalar variable from first principle,
• find the derivative of sums, scalar products and products of vectors;
• find equations for the tangent, principal norma and binormal;
• determine the curvature, radius of curvature and center of curvature of any curve with a given position, and vector.
• define and work examples on indefinite integral and definite integral of vector functions of a given variable.

Pre-Test (See Post-Test.

Differentiation of a vector with respect to a scalar (from first principle)

Supposing both the magnitude and the direction of a vector $\vec{v}$ depend upon the value of a scalar variable t, both may change, for instance, with the time. If we denote this dependence upon t by writing $\vec{v} = \vec{v}(t)$, then the differential coefficient, or derivative, of $\vec{v}$ with respect to t is defined as a limiting process by the equation

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \lim_{t\to 0} \frac{\vec{v}(t+\delta t) - \vec{v}(t)}{\delta t} \\ &= \lim_{t\to 0} \frac{\delta \vec{v}}{\delta t}. \end{aligned}$$

In Figure 26 below,

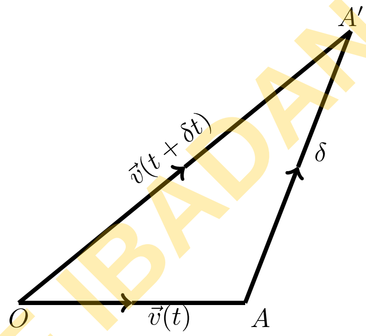


Figure 26:

if $\vec{v}(t)$ and $\vec{v}(t+\delta t)$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OA'}$ respectively, then $\delta\vec{v}$ is represented by $\overrightarrow{AA'}$.

Since the division of a vector by a scalar does not destroy its vector nature, the derivative of a vector with respect to a scalar variable is itself a vector.

The fact is perhaps best demonstrated by writing $\vec{v}$ in terms of its components,

$$\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

here (a_1, a_2, a_3) are scalar variables of t, while $(\vec{i}, \vec{j}, \vec{k})$ are unit vectors having fixed directions in space. If $(\delta a_1, \delta a_2, \delta a_3)$ now represent the changes in (a_1, a_2, a_3) consequent upon a small change δt in t, we have

$$\delta\vec{v} = \delta a_1\vec{i} + \delta a_2\vec{j} + \delta a_3\vec{k},$$

which, on dividing by δt and proceeding to the limit, gives

$$\frac{d\vec{v}}{dt} = \frac{da_1}{dt}\vec{i} + \frac{da_2}{dt}\vec{j} + \frac{da_3}{dt}\vec{k}.$$

Thus, $\dfrac{d\vec{v}}{dt}$ is clearly a vector.

The following results may now be deduced.

(i) $\dfrac{d}{dt}(\vec{v}+\vec{w}) = \dfrac{d\vec{v}}{dt} + \dfrac{d\vec{w}}{dt}$.

For, expressing $\vec{v}+\vec{w}$ in terms of its components,

$$\begin{aligned}\vec{v}+\vec{w} &= (a_1+b_1)\vec{i} + (a_2+b_2)\vec{j} + (a_3+b_3)\vec{k} \\ &= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) + (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}).\end{aligned}$$

On differentiating, we thus have

$$\begin{aligned}\frac{d}{dt}(\vec{v}+\vec{w}) &= \left(\frac{da_1}{dt}\vec{i} + \frac{da_2}{dt}\vec{j} + \frac{da_3}{dt}\vec{k}\right) + \left(\frac{db_1}{dt}\vec{i} + \frac{db_2}{dt}\vec{j} + \frac{db_3}{dt}\vec{k}\right) \\ &= \frac{d\vec{v}}{dt} + \frac{d\vec{w}}{dt}.\end{aligned}$$

(ii) $\dfrac{d}{dt}(\vec{v}\cdot\vec{w}) = \vec{v}\cdot\dfrac{d\vec{w}}{dt} + \dfrac{d\vec{v}}{dt}\cdot\vec{w}$, we have

$$\vec{v}\cdot\vec{w} = a_1b_1 + a_2b_2 + a_3b_3;$$

Therefore, differentiating

$$\begin{aligned}\frac{d}{dt}(\vec{v}\cdot\vec{w}) &= \left(a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right) + \left(\frac{da_1}{dt}b_1 + \frac{da_2}{dt}b_2 + \frac{da_3}{dt}b_3\right) \\ &= \vec{v}\cdot\frac{d\vec{w}}{dt} + \frac{d\vec{v}}{dt}\cdot\vec{w}.\end{aligned}$$

Since both terms on the right-hand side are scalar products, it follows that the order in which the vectors occur in the respective products is immaterial.

(iii) $\dfrac{d}{dt}(\vec{v}\times\vec{w}) = \vec{v}\times\dfrac{d\vec{w}}{dt} + \dfrac{d\vec{v}}{dt}\times\vec{w}$

Writing $\vec{v}\times\vec{w}$ in terms of its components

$$\vec{v}\times\vec{w} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_3b_2 - a_2b_3)\vec{k},$$

therefore,

$$\begin{aligned}\frac{d}{dt}(\vec{v}\times\vec{w}) &= \left(a_2\frac{db_3}{dt} - a_3\frac{db_2}{dt}\right)\vec{i} + \left(a_3\frac{db_1}{dt} - a_1\frac{db_3}{dt}\right)\vec{j} \\ &\quad + \left(a_1\frac{db_2}{dt} - a_2\frac{db_1}{dt}\right)\vec{k} + \left(\frac{da_2}{dt}b_3 - \frac{da_3}{dt}b_2\right)\vec{i} \\ &\quad + \left(\frac{da_3}{dt}b_1 - \frac{da_1}{dt}b_3\right)\vec{j} + \left(\frac{da_1}{dt}b_2 - \frac{da_2}{dt}b_1\right)\vec{k} \\ &= \vec{v}\times\frac{d\vec{w}}{dt} + \frac{d\vec{v}}{dt}\times\vec{w}.\end{aligned}$$

Since a vector product changes its sign if the order in which the vectors occur in it is inverted, it follows that the order in which $\vec{v}, \vec{w}$ and their derivatives occur in the above vector products must be maintained as shown.

Practice Exercise

If a vector $\vec{v}$ is written as the product of its module a with the direction vector $\vec{n}$, whose module is unity. Show that

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(a\vec{n}) = \frac{da}{dt}\vec{n} + a\frac{d\vec{n}}{dt}$$

Some special results relating to the rate of change of a vector, and which may be readily deduced form the differential properties derived above, are important.

(1) If $\vec{v}$ is a vector of constant magnitude, then we have

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}(a^2) = 0.$$

But, from (ii), $\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 2\cdot\frac{d\vec{v}}{dt}$ so that

$$\vec{v}\cdot\frac{d\vec{v}}{dt} = 0,$$

i.e. $\frac{d\vec{v}}{dt}$ must be perpendicular to $\vec{v}$.

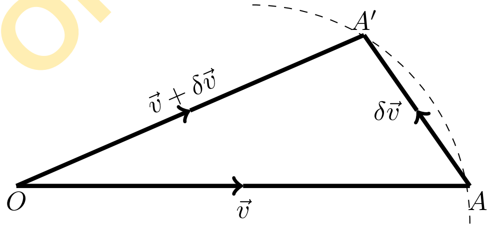


Figure 27:

This result might also have been deduced quite simply in another way.

In Figure 27 above, let $\overrightarrow{OA}$ and $\overrightarrow{OA'}$ represent the vectors $\vec{v}$ and $\vec{v}+\delta\vec{v}$ corresponding to times t and $t+\delta t$,respectively; then, since $\vec{v}$ is a vector of constant magnitude, $\vec{v}$ and $\vec{v}+\delta\vec{v}$ have the same lengths, so that

$$OA = OA'$$

Consequently, as $\delta\vec{v} \to 0$, A' approaches A along the arc of a spherical curve having 0 as centre. The limiting direction of $\dfrac{d\vec{v}}{dt}$ is thus along the tangent to the curve at A, and so is at right angles to $\vec{v}$.

(2) From (iv) above, we have

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(a\vec{n}) = \frac{da}{dt}\vec{n} + a\frac{d\vec{n}}{dt}.$$

But since $|\vec{n}|$ is a unit vector, $\vec{n}$ and $\dfrac{d\vec{n}}{dt}$ are orthogonal vectors.
Consequently, we have

$$\left|\frac{d\vec{v}}{dt}\right|^2 = \left(\frac{da}{dt}\right)^2 + a^2\left|\frac{d\vec{n}}{dt}\right|^2,$$

from which it follows that $\left|\dfrac{d\vec{v}}{dt}\right|$ is not in general equal to $\dfrac{da}{dt}$, or, in words, the module of the rate of change of a vector is not in general equal to the rate of change of its module.

Example

Consider the motion of a point P describing a circle of radius a about a centre O. Writing $\overrightarrow{OP} = a$, the velocity of P in space is given by

$$\vec{v} = \frac{da}{dt},$$

and hence

$$\left|\frac{da}{dt}\right|^2 = \vec{v}^2 = a^2\vec{w}^2,$$

where $\vec{w}$ is the angular velocity of P in the circle.

But, for circular motion, a is constant, so that $\dfrac{da}{dt} = 0$.

Consequently,

$$\left|\frac{da}{dt}\right| \neq \frac{da}{dt}.$$

(3) **Curvature:** Consider the Figure 28 below.

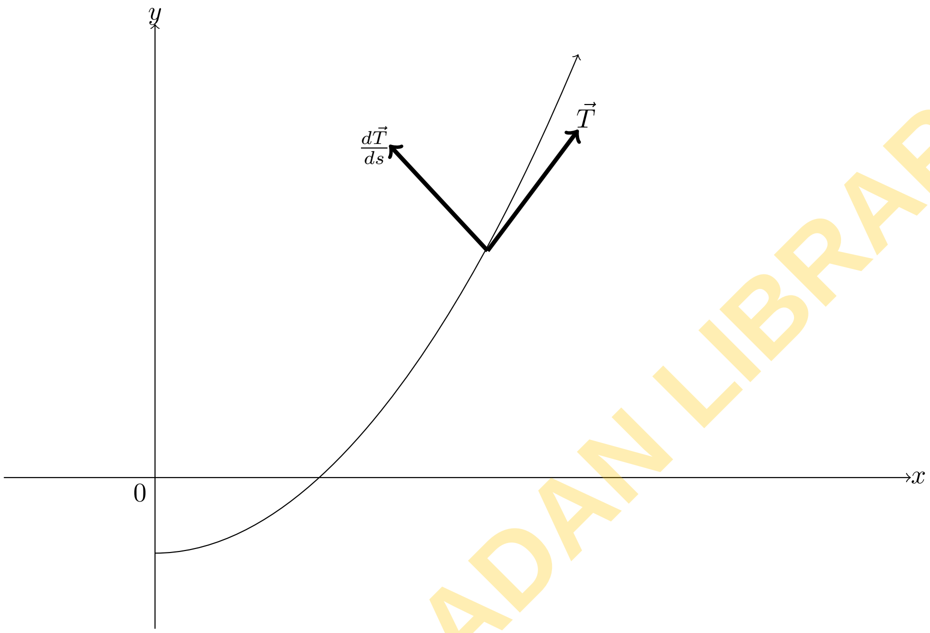


Figure 28:

If $\overrightarrow{T}$ is the unit tangent vector to the space curve Γ at P (see Figure 28 above), $|\overrightarrow{T}| = 1$ and $\dfrac{d\overrightarrow{T}}{ds}$ is perpendicular to $\overrightarrow{T}$.

A directed line through P in the direction of $\dfrac{d\overrightarrow{T}}{ds}$ is therefore a normal of the curve, it is called the principal normal of Γ at P. If $\overrightarrow{N}$ is a unit vector in the direction of the principal normal, we may write $\dfrac{d\overrightarrow{T}}{ds} = k\overrightarrow{N}$ where $k = \left|\dfrac{d\overrightarrow{T}}{ds}\right|$. The positive number k is called the curvature of Γ at P.

Practise Exercise:

Show that the curvature of a circle is everywhere equal to the reciprocal of its radius.

Let us return now to our general curve Γ whose curvature at P is k. This is also the curvature of a circle of radius $\dfrac{1}{k}$; for this reason $\dfrac{1}{k}$ is called the radius of curvature of Γ at P.

If we denote the radius of curvature by ρ, $\rho = \dfrac{1}{k}$.

The center of curvature of Γ at P is a point C on the principal normal at a distance ρ from C in the direction of $\overrightarrow{N}$; $\overrightarrow{PC} = \rho\overrightarrow{N}$ and $\overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PC} = r + \rho\overrightarrow{N}$.
A directed line through P in the direction of the unit vector $\overrightarrow{B} = \overrightarrow{T} \times \overrightarrow{N}$ is called the binormal of Γ at P. The binormal is thus perpendicular to both tangent and principal normal at P.

At every point of a curve the three unit vectors $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}$ form a right-handed set giving the positive directions along the tangent, principal normal, and binormal, respectively. Therefore,

$$\overrightarrow{T} \times \overrightarrow{N} = \overrightarrow{B}, \quad \overrightarrow{N} \times \overrightarrow{B} = \overrightarrow{T}, \quad \overrightarrow{B} \times \overrightarrow{T} = \overrightarrow{N}.$$

Integration of a vector function

If $\vec{f}(t)$ and $\overrightarrow{F}(t)$ are vector functions of the variable t such that

$$\frac{d\overrightarrow{F}}{dt} = \vec{f}, \tag{1}$$

then $\overrightarrow{F}$ is called an integral of $\vec{f}$ with respect to t and is written

$$\overrightarrow{F} = \int \vec{f}\,dt \tag{2}$$

In other words, both (1) and (2) state the same fact: the derivative of $\overrightarrow{F}$ with respect to t is u. The process of finding a function which has a given derivative is called integration. In (2), the function $\vec{f}$ - the integrated - is integrated to produce $\overrightarrow{F}$. If $\overrightarrow{F}$ is any function which satisfies (1), then $\overrightarrow{F} + \vec{c}$, where $\vec{c}$ is an arbitrary constant vector, will also satisfy (1). Hence the integral $\overrightarrow{F}$ is indefinite to the extent of an additive vector constant - the constant of integration. For this reason the function $\overrightarrow{F}$ denoted by (2) is called an indefinite integral.

Just as in the calculus we may show that

$$\int (\vec{u} + \vec{w})dt = \int \vec{u}\,dt + \int \vec{w}\,dt;$$

$$\int c\vec{u}\,dt = c\int \vec{u}\,dt, \ \int c\vec{u}\,dt = c\int \vec{u}\,dt$$

If $\vec{u}$ is expressed in terms of its components, we have for $\vec{u} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$,

$$\int \vec{u}\,dt = \vec{i}\int a_1\,dt + \vec{j}\int a_2\,dt + \vec{k}\int a_3\,dt.$$

The integration of vector functions is thus reduced to the integration of scalar functions.

Definition (Definite Integration)

The definite integral $\int_a^b \vec{u}(t)dt$ of the vector function $\vec{u}(t)$ is defined as a limit of a sum just as in the ordinary calculus. If $\vec{u}(t)$ is expressed in terms of its components, we have,

$$\int_a^b \vec{u}\ dt = \vec{i}\int_a^b a_1\ dt + \vec{j}\int_a^b a_2\ dt + \vec{k}\int_a^b a_3\ dt.$$

Thus vector definite integrals may be reduced to scalar definite integrals.

Owing to this fact many of the properties of scalar definite integrals may be extended at once to the vector case; for example

$$\int_a^b \vec{u}\ dt = -\int_b^a \vec{u}\ dt.$$

Moreover, if $\overrightarrow{F}(t)$ is an indefinite integral of $\vec{f}(t)$ we have the fundamental result

$$\int_a^b \vec{f}(t)dt = \overrightarrow{F}(b) - \overrightarrow{F}(a) \tag{3}$$

If the upper limit in (3) is the variable of integration,

$$\int_a^b \vec{f}(t)dt = \overrightarrow{F}(t) - \overrightarrow{F}(a).$$

and hence

$$\frac{d}{dt}\int_a^t \vec{f}(t)dt = \frac{d\overrightarrow{F}(t)}{dt} = \vec{f}(t) \tag{4}$$

Summary

The derivative $\dfrac{d\vec{u}}{dt}$ of a vector $\vec{u}$ with respect to a scalar variable t is defined as the limit of $\dfrac{\delta\vec{u}}{\delta t}$ as δt approaches zero.

If $\vec{u} = \overrightarrow{OP}$ and P describes the curve γ when O is held fast, $\dfrac{d\vec{u}}{dt}$ is a vector tangent to Γ at P in the direction of increasing t. In particular, if the magnitude of $\vec{u}$ is constant, $\dfrac{d\vec{u}}{dt}$ is perpendicular to $\vec{u}$.

The derivative of a constant vector is zero. The derivative of the sum $\vec{u}+\vec{w}$ and the products $f\vec{u}$, $\vec{u}\cdot\vec{w}$, $\vec{u}\times\vec{w}$ are found by rules of the same form as those given in the Calculus for differentiating a sum and a product, in the case of $\vec{u}\times\vec{v}$, however, the order of the factors must be preserved. Moreover, the rule for differentiating a function of a function is like that in the Calculus.

At any point P of a curve the derivative of the unit tangent vector with respect to the arc $\left(\dfrac{d\vec{T}}{ds}\right)$ gives the direction of the principal normal; and the numerical value k of this derivative is called the curvature at P. The reciprocal of the curvature, $\rho = \dfrac{1}{k}$, is called the radius of curvature. The point C given by $\overrightarrow{PC} = \rho\vec{N}$ is the center of curvature.

The indefinite integral of $\vec{f}(t)$ is any function $\vec{F}(t)$ such that $\dfrac{d\vec{F}}{dt} = \vec{f}$. The definite integral $\displaystyle\int_a^b \vec{f}(t)dt$ is defined as the limit of a sum just as in the scalar Calculus; it equal $\vec{F}(b) - \vec{F}(a)$.

Post-Test

1. If $\vec{r}$ is the position vector along a plane curve Γ, $\vec{r}' = \vec{r} + c\vec{N}$ is the position vector along a parallel curve Γ' at a normal distance c from Γ. Prove that

$$T'\frac{ds'}{ds} = (1-c_k)T$$

and hence

$$T' = T, \quad \frac{ds'}{ds} = 1 - c_k.$$

Show also that $\rho' = \rho - c$ and $s' = s = c\psi$ when both s and s' are measured from a common normal.

2. If $\vec{v} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{w} = (2t-3)\vec{i} + \vec{j} - t\vec{k}$, find (a) $\dfrac{d}{dt}(\vec{v}\cdot\vec{w})$, (b) $\frac{d}{dt}(\vec{v}\times\vec{w})$,

(c) $\frac{d}{dt}|\vec{v}+\vec{w}|$, (d) $\dfrac{d}{dt}\left(\vec{v}\times\dfrac{d\vec{w}}{dt}\right)$ at $t=1$.

3. Find a unit tangent vector to any point on the curve $x = a\cos wt$, $y = a\sin wt$, $z = bt$ where a, b, w are constants.

4. Show that the radius of curvature of a plane curve with equations $y = f(x)$, $z = 0$, that is, a curve in the xy-plane is given by

$$\rho = \frac{[1+(y')^2]^{3/2}}{|y''|}.$$

5. Find the curvature and radius of curvature of the curve with position vector $\vec{r} = a\cos u\ \vec{i} + b\sin u\vec{j}$, where a and b are positive constants. Interpret the case where $a = b$.

6. Find equation for the (a) tangent, (b) principal normal and (c) binormal to the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ at the point where $t = \pi$.

7. If $\vec{v}(t) = t\vec{i} - t^2\vec{j} + (t-1)\vec{k}$ and $\vec{w}(t) = 2t^2\vec{i} + 6 + \vec{j}$, evaluate

   (a) $\displaystyle\int_0^2 \vec{v}\cdot\vec{w}\,dt$

   (b) $\displaystyle\int_0^2 \vec{v}\times\vec{w}\,dt$

8. If $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$, evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ along the curve C in the xy-plane, $y = x^3$ from the point $(1,1)$ to $(2,8)$.

Supplementary Reading

1. M.R. Spiegel, Theory and Problems of Vector Analysis, Schaum Publishing Co., New York (1959).

2. P. Tipler, Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

LECTURE 6

Equations of a Line and a Plane

Introduction

In this lecture, we shall derive another property of vectors called the vector equation of a line and the vector equation of a plane.

Objective
At the end of this lecture you should be able to:
- find the distance between a point and a plane,
- find the angle between any two given planes,
- determine a vector parallel to the line of intersection of any given two planes;
- prove that three points A, B, C are collinear if and only if $\overrightarrow{AC} \times \overrightarrow{AB} = 0$, and
- prove that four points A, B, C, D are coplanar if and only if $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0$.

Pre-Test: (See Post-Test)

Lines

Suppose L is a line in space that passes through a given point $P_1(x_1, y_1, z_1)$ and is parallel to a given non-zero vector

$$\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

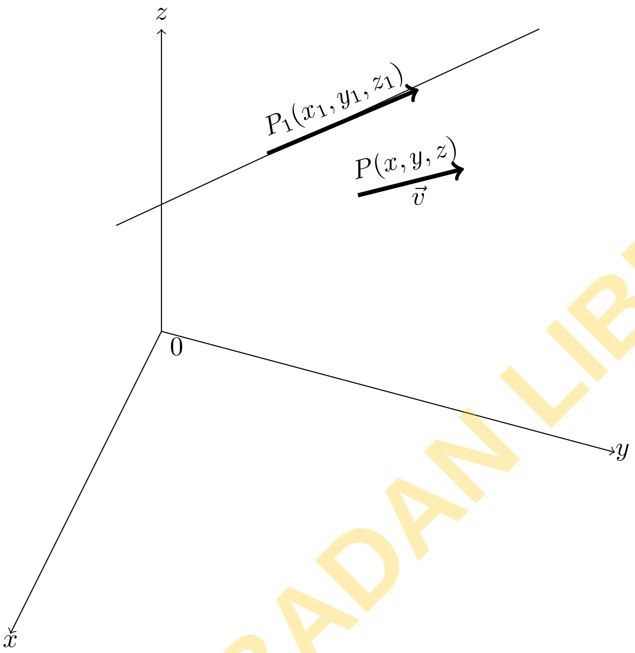


Figure 29:

Then L is the set of all points $P(x, y, z)$ for which the vector $\overrightarrow{P_1P}$ is parallel to the given vector $\vec{v}$ (see Figure 29 above).
That is, P is on the line L if and only if there is a scalar t such that

$$\overrightarrow{P_1P} t\vec{v} \tag{1}$$

When we separate the components in eqn (1), we have

$$x - x_1 = ta_1, \;\; y - y_1 = ta_2, \;\; z - z_1 = ta_2 \tag{2}$$

We may eliminate t from the equations in (2) to obtain the following

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3} \tag{3}$$

If any a_1, a_2 or a_3 is zero in eqn (3), then the corresponding numerator is also zero.
For example,

$$x - x_1 = ta_1 \;\text{ and }\; a_1 = 0$$

together imply that

$$x - x_1 = 0.$$

Practise Exercise

- Determine K so that P is the midpoint of the line segment $\overrightarrow{AB}$
- If $\overrightarrow{AP} = -\frac{2}{3}\overrightarrow{AB}$, locate P relative to A and B.
- Interpret the equation $r = \frac{2}{3}r_1 + \frac{1}{3}r_2$.

Planes

To obtain an equation for a plane, we suppose that a point $P_1(x_1, y_1, z_1)$ on the plane and a nonzero vector

$$\vec{n} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \tag{4}$$

Perpendicular to the plane are given. Then the point $P(x, y, z)$ will lie in the plane if and only if

$$\vec{n} \cdot \overrightarrow{p_1p} = 0$$

or

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0 \tag{5}$$

This equation may also be put in the form

$$a_1x + a_2y + a_2z = a_4 \tag{6}$$

where a_4 is the constant $a_1x_1 + a_2y_1 + a_3z_1$.
Conversely, if we start from any linear equation such as (6), we may find a point $P_1(x_1, y_1, z_1)$ whose coordinates do satisfy it; that is, such that

$$a_1x_1 + a_2y_1 + a_3z_1 = a_4$$

Then by subtraction, we may put the given equation (6) into the form eqn (5) and factor it into the dot product.

$$\vec{n} \cdot \overrightarrow{P_1P} = 0,$$

with $\vec{n}$ as in eqn (4).
This says that the constant vector $\vec{n}$ is perpendicular to the vector $\overrightarrow{P_1P}$ for every pair of points P_1 and P whose coordinates satisfy such a linear equation is a plane and the vector $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, with the same coefficients that x, y and z have in the given equation, is normal to the plane.

Example 1

Find the distance d between the points $P(2, -3, 4)$ and the plane $x + 2y + 2z = 13$.

Solution

Carry out the following steps.

1st Find a line L through P normal to the plane.

2nd Find the coordinates of the point in which the line meets the plane.

3rd Compute the distance between P and Q.

The vector $\vec{n} = \vec{i} + 2\vec{j} + 2\vec{k}$ is normal to the given plane, and the line

$$L : \frac{x-2}{1} = \frac{y+3}{2} = \frac{z-4}{2}$$

goes through P and is parallel to $\vec{n}$. Hence, L is normal to the plane.
If we denote the common ratio in the equations for L by t,

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-4}{2} = t$$

we have

$$x = t+2, \quad y = 2t-3, \quad z = 2t+4$$

as parametric equations of the line in terms of the parameter t. Substituting these into the equation of the plane, we obtain

$$(t+2) + 2(2t-3) + 2(2t+4) = 13,$$

or $t = 1$ at the point of intersection of the plane and the line L. That is, $Q(3, -1, 6)$ is the point of intersection.
The distance between the point and the plane is the distance between $P(2, -3, 4)$ and $Q(3, -1, 6)$. Hence

$$d = \sqrt{(3-2)^2 + (-1+3)^2 + (6-4)^2} = 3.$$

Example 2

Find the angle between the two planes

$$3x - 6y - 2z = 7 \text{ and } 2x + 2y - 2z = 5.$$

Solution

Note that the angle between two planes can be obtained from their normals.
Clearly the angle between two planes is the same as the angle between their normals. (Actually there are two angles in each case, namely, θ and $180^0 - \theta$.)
From the equations of the planes we may read off their normal vectors:

$$\vec{n_1} = 3\vec{i} - 6\vec{j} - 2\vec{k}, \quad \vec{n_2} = 2\vec{i} + \vec{j} - 2\vec{k}$$

Then,

$$\begin{aligned} \cos\theta &= \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{4}{21}, \\ \theta &= \cos^{-1}\left(\frac{4}{21}\right) \simeq 79^0. \end{aligned}$$

Example 3

Find an equation of the plane that passes through the two points $P_1(1,0,-1)$ and $P_2(-1,2,1)$ and is parallel to the line of intersection of the planes $3x+y-2z=6$ and $4x-y+3z=0$.

Solution

The coordinates of either one of the points P_1 or P_2 will do for the x_1, x_2 and x_3 in eqn (5). What remains, then, is to find a vector $\vec{n}$ normal to the plane in question to furnish the coefficients a_1, a_2 and a_3 of eqn (5).
The line of intersection of the two given planes is parallel to the vector

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 4 & -1 & 3 \end{vmatrix} = \vec{i} - 17\vec{j} - 7\vec{k},$$

where $\vec{n_1}$ and $\vec{n_2}$ are normals to the two given planes. The vector

$$\overrightarrow{P_1P_2} = -2\vec{i} + 2\vec{j} + 2\vec{k}$$

is to line in the required plane.

Now we may slide $\vec{v}$ parallel to itself until it also lies in the required plane (since the plane is to be parallel to $\vec{v}$). Hence, we may take

$$\vec{n} = \overrightarrow{P_1P_2} \times \vec{v} = 20\vec{i} - 12\vec{j} + 32\vec{k}.$$

as a vector normal to the plane.
Actually, $\frac{1}{4}\vec{n} = 5\vec{i} - 3\vec{j} + 8\vec{k}$ serves just as well. From this normal vector, we may substitute

$$a_1 = 5, \quad a_2 = -3, \quad a_3 = 8$$

in eqn (5), together with $x_1 = 1$, $y_1 = 0$, $z_1 = -1$, since $P_1(1,0,-1)$ is to lie in the plane. The required plane is therefore

$$5(x-1) - 3(y-0) + 8(z+1) = 0$$

or

$$5x - 3y + 8z + 3 = 0.$$

Summary

Given a point $P_1(x_1, y_1, z_1)$ and a non zero vector $\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.
$L = \{P(x,y,z) : \overrightarrow{P_1P}$ is parallel to $\vec{v}\}$
or
$L = \{P(x,y,z) : \exists$ a scalar t such that $\overrightarrow{P_1P} = t\vec{v}\}$.
The Cartesian equations for the line L:
$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}$.
Given a point $P_1(x_1, y_1, z_1)$ on the plane and a nonzero vector
$\vec{n} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$
Perpendicular to the plane, then we have $P(x,y,z)$ lies in the plane through P_1 perpendicular to $\vec{n}$ if and only if
$\overrightarrow{P_1P} \cdot \vec{n} = 0$.
That is,
$a_1x + a_2y + a_3z = a_4$ where $a_4 = a_1x_1 + a_2y_1 + a_3z_1$ is a constant.

Post-Test

1. Find the coordinates of the point P in which the line

$$\frac{x-1}{2} = \frac{y+1}{-1} = -\frac{z}{3}$$

intersects the plane $3x + 2y - z = 5$.

2. Find parametric and Cartesian equations of the line joining the points $A(1, -2, -1)$ and $B(-1, 0, 1)$.

3. Show , by vector methods, that the distance from the point $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz - D = 0$ is

$$\frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

4(a) What is meant by the angle between a line and a plane?

(b) Find the acute angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$$

and the plane $10x + 2y - 11z = 3$.

5. Find a plane that passes through the point $(1, -1, 3)$ and is parallel to the plane $3x + y + z = 7$.

6. Prove that the line

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{4}$$

is a parallel to the plane

$$x - 2y + z = 6.$$

7(a) Prove that three points A, B, C are collinear if and only if $\overrightarrow{AC} \times \overrightarrow{AB} = 0$.

(b) Are the points $A(1, 2, -3), B(3, 1, 0), C(-3, 4, -9)$ collinear?

8. Prove that four points A, B, C, D are coplanar if and only if $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0$.

9. Show that the line of intersection of the planes $x + 2y - 2z = 5$ and $5x - 2y - z = 0$ is parallel to the line

$$\frac{x+3}{2} = \frac{y}{3} = \frac{z-1}{4}.$$

Find the plane determined by these two lines.

10. Find the direction cosines of the line $2x + y - z = 5$, $x - 3y + 2z = 2$.

Supplementary Reading Supplementary Reading

1. M.R. Spiegel, Theory and Problems of Vector Analysis, Schaum Publishing Co., New York (1959).

2. P. Tipler, Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

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4. L.J. Meriam and L.G. Kraige, Engineering Mechanics: Statics and Dynamics, 5th edition, John Wiley & Sons, Inc., New York, NY, 2002.

5. F.P. Beer and E.R. Johnston, Vector Mechanics for Engineers (Statics). McGraw-Hill, 3rd Edition (1999).

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LECTURE 7

General Kinematics

Introduction

Kinematics is that division of mechanics which treats the geometry of motion of bodies, without taking into account their inertia (mass) or the forces acting on them.

The principal problem of kinematics is that of determining all the kinematics characteristics of the motion of a body as a whole or of any of its particles (path, velocity, acceleration, etc.) when the law of motion for the given body is known.

For the solution of this problem we must know either the equations of motion for the given body or for another body kinematically associated with it.

In this lecture, we shall start the study of kinematics with an investigation of the motion of the simplest body - a particle (kinematics of a particle), proceeding later to the examination of the kinematics of rigid bodies. We shall also consider the cases of rectilinear motion of a particle, of curvilinear motion of a particle, of translatory and rotational motion of a rigid body, of plane motion of a rigid body, of motion of a rigid body having one fixed point and motion of a free rigid body, of resultant motion of a particle, and of resultant motion of a rigid body.

Objectives
At the end of this lecture you should be able to:
- analyze the kinematics of particles and rigid bodies in motion;
- determine the velocity and acceleration of a point using Cartesian, polar, and path coordinates; use coordinate transformations to change the component definitions of a vector in moving from one system to another.
- recognize the need for kinematic constraints in developing equations of motion.
- develop general kinematic equations for planar mechanisms, and
- derive equations of motion for planar mechanisms via free-body diagrams.

Pre-Test (See Post-Test)

Linear motion

Linear or translational kinematics is the description of the motion in space of a point along a line, also known as trajectory or path. This path can be either straight (rectilinear) or curved (curvilinear). There are three basic concepts that are required for understanding linear motion:

Displacement (denoted by $\vec{r}$ below) is the "vector" version of distance and direction. It is the shortest distance between two point locations.

Relative to some origin, (say at $0 = (0,0,0)$) using a coordinate system defined by the observer, the two points might be at $\vec{r_1}$ and $\vec{r_2}$. Because displacement is a vector, the displacement between the two points is found by vector subtraction as

$$\vec{r} = \vec{r_1} - \vec{r_2}$$

Velocity (denoted by $\vec{v}$ below) is the measure of the rate of change in displacement with respect to time (m/sec.) that is the displacement of a point changes with time. Velocity also is a vector. Instantaneous velocity (the velocity at an instant or time) is defined as

$$\vec{v} = \frac{d\vec{r}}{dt},$$

where $d\vec{r}$ is an infinitesimally small displacement and dt is an infinitesimally small length of time.

Because $d\vec{r}$ is necessarily the distance between two infinitesmally space points along the trajectory of the point, it is the same as an increment in arc length along the path of the point, customarily denoted $d\vec{s}$. Average velocity (velocity over a length of time) is defined as

$$\vec{v}_{\text{ave}} = \frac{\triangle\vec{r}}{\triangle t},$$

where $\triangle\vec{r}$ is the change in displacement and $\triangle t$ is the interval of time over which displacement changes. As $\triangle t$ becomes smaller and smaller, $\vec{v}_{\text{ave}} \longrightarrow \vec{v}$.

For a velocity constant in magnitude and direction, every unit of time adds the length of the velocity vector (in the same direction) to the displacement of the moving point. If the change in displacement (a vector) is known, the velocity is parallel to it.

Acceleration (denoted by $\vec{a}$ below) is the vector quantity describing the rate of change with time of velocity. Acceleration is also a vector. Instantaneous acceleration (the acceleration at an instant of time) is defined as:

$$\vec{a} = \frac{d\vec{v}}{dt},$$

where $d\vec{v}$ is an infinitesimally small change in velocity and dt is an infinitesimally small length of time. Average acceleration (acceleration over a length of time) is defined as:

$$\vec{v}_{\text{ave}} = \frac{\triangle\vec{v}}{\triangle t},$$

where $\triangle\vec{v}$ is the change in velocity and $\triangle t$ is the interval of time over which velocity changes. As $\triangle t$ becomes smaller and smaller,

$$\vec{v}_{\text{ave}} \longrightarrow \vec{a}.$$

If acceleration is constant in magnitude and direction, for every unit of time the length of the acceleration vector (in the same direction) is added to the velocity. If the change in velocity (a vector) is known, the acceleration is parallel to it.

Types of motion

There are two types of motion in general:
uniform and non-uniform.
uniform motion implies constant velocity in a straight line.
Non-uniform motion implies acceleration. If the acceleration changes in time, the rate of change of acceleration is called the jerk.

Integral relations

The above definitions can be inverted by integration to find:

$$\begin{aligned}
\vec{v}(t) &= \vec{v_0} + \int_0^t dt'\vec{a}(t') \\
\vec{r}(t) &= \vec{r_0} + \int_0^t dt'\vec{v}(t') \\
&= \vec{r_0} + \vec{v_0}t + \int_0^t dt' \int_0^{t'} dt''\vec{a}(t'') \\
&= \vec{r_0} + \vec{v_0}t + \int_0^t dt'(t-t')\vec{a}(t'),
\end{aligned}$$

where the double integration is reduced to one integration by interchanging the order of integration, and subscript 0 signifies evaluation at $t = 0$ (initial values).

Constant acceleration

Integrating acceleration ($\vec{a}$) with respect to time (t) gives the change in velocity. When acceleration is constant both n direction and in magnitude, the point is said to be undergoing uniformly acceleration motion. In this case, the above equations can be simplified:

Eq. (1) $\vec{v} = \int_0^t \vec{a}dt' = \vec{v}_0 + \vec{a}t$

Those who are familiar with calculus may recognize this as an initial values problem. When the acceleration is constant, simple addition of the product of acceleration and time to the initial velocity ($\vec{v}_0$) gives the final velocity $\vec{v}$).

Another time integration provides the displacement of the object, assuming an initial position at time $t = 0$ of $\vec{r}_0$.

Eq. (2) $\vec{r} = \vec{r}_0 + \int_0^t \vec{v}dt' = \vec{r}_0 + \int_0^t (\vec{v}_0 + \vec{a}t)dt'$

$$= \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

Using the above formula, we can substitute for $\vec{v}$ to arrive at the following equation, where $\vec{r}$ is displacement.

Eq. (3) $\vec{r} = \vec{r}_0 + \frac{\vec{v}+\vec{v}_0 t}{2}$.

By using the definition of an average, this equation states that when the acceleration is constant average velocity times time equals displacement.

For convenience, set $\vec{r}_0 = 0$. Using Eq. (1) to find $\vec{v} - \vec{v}_0$ and multiply by Eq.(3) we fine a connection between the final velocity at time t and the displacement at that time:

$$\vec{r} \cdot \vec{a}t = (\vec{v} - \vec{v}_0) \cdot \frac{\vec{v} + \vec{v}_0 t}{2},$$

where the "$\cdot$" denotes a vector dot product.
Dividing the t on both sides and carrying out the dot products:

Eq. (4) $2\vec{r} \cdot \vec{a} = v^2 - v_0^2$.

For the case where $\vec{r}$ is parallel to $\vec{a}$ resulting in a straight-line motion, the vector $\vec{r}$ has magnitude equal to the path length s at time t, and this equation becomes:

$$v^2 = v_0^2 + 2as,$$

which can be a useful result when time is not known explicitly.

Relative velocity

To describe the motion of object A with respect to object B, when we know how each is moving with respect to a reference object O, we can use vector algebra. Choose an origin for reference, and let the position of objects A, B, and O be denoted by $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_O$. Then the position of A relative to the reference object O is

$$\vec{r}_{A/O} = \vec{r}_A - \vec{r}_O$$

Consequently, the position of A relative to B is

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{r}_A - \vec{r}_O - (\vec{r}_B - \vec{r}_O) = \vec{r}_{A/O} - \vec{r}_{B/O}$$

The above relative equation states that the motion of A relative to B is equal to the motion of A relative to O minus the motion of B relative to O. It may be easier to visualize this result if the terms are re-arranged:

$$\vec{r}_{A/} = \vec{r}_{A/B} + \vec{r}_{B/O},$$

or, in words, the motion of A relative to the reference is that of B plus the relative motion of A with respect to B. These relations between displacements become relations between velocities by simple time-differentiation, and a second differentiation makes them apply to accelerations.

For example,

$$\overrightarrow{V}_A = \overrightarrow{V}_B + \overrightarrow{V}_{A/B}$$
$$\overrightarrow{V}_{A/B} = \overrightarrow{V}_A - \overrightarrow{V}_B$$

At velocities comparable to the speed of light, these equations are not valid. They are replaced by equations derived from Einstein's theory of special relativity.

$$\vec{a}_A = \vec{a}_B + \vec{a}_{A/B}$$

To find $\vec{a}_{A/B}$ we simply rearrange this equation to obtain:

$$\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B.$$

Example: <u>Rectilinear 1-dimensional motion</u>

Consider an object that is fired directly upwards and falls back to the ground so that its trajectory is contained in a straight line. If we adopt the convention that the upward direction is the positive direction, the object experiences a constant acceleration of approximately - 9.81 m/s^2. Therefore, its motion can be modeled with the equations governing uniform accelerated motion.

For the sake of example, assume the object has an initial velocity of +50 m/s. There are several interesting kinematic questions we can ask about the particle's motion.

How long will it be airborne?

To answer this question, we apply the formula

$$\vec{x}_f - \vec{x}_i = \vec{v}_i t + \frac{1}{2}\vec{a}t^2.$$

Since the question asks for the length of time between the object leaving the ground and hitting the ground on its fall, the displacement is zero.

$$0 = \vec{v}_i t + \frac{1}{2}\vec{a}t^2 = t\left(\vec{v}_i + \frac{1}{2}\vec{a}t\right).$$

We find two solutions for it. The trivial solution says the time is zero; this is actually also true, it is the first moment the displacement is zero: just when it starts motion. However, the solution of interest is

$$t = -\frac{2\vec{v}_i}{\vec{a}} = -\frac{2 \times 50}{-9.81} = 10.2\ s.$$

What altitude will it reach before it begins to fall?
In this case, we use the fact that the object has a velocity of zero at the apex of its trajectory. Therefore, the applicable equation is:

$$\vec{v}_f^2 = \vec{v}_i^2 + 2\vec{a}(\vec{x}_f - \vec{x}_i)$$

If the origin of our coordinate system is at the ground, then $\vec{x}_i$ is zero. Then we solve for $\vec{x}_f$ and substitute known values:

$$\vec{x}_f = \frac{\vec{v}_f^2 - \vec{v}_i^2}{2\vec{a}} + \vec{x}_i = \frac{0 - 50^2}{2 \times (-9.81)} + 0 = 127.55\ m$$

What will its final velocity be when it reaches the ground?
To answer this question, we use the fact that the object has an initial velocity of zero at the apex before it begins its descent. We can use the same equation we used for the last question, using the value of 127.55 m for $\vec{x}_i$

$$\vec{v}_f = \sqrt{\vec{v}_i^2 + (\vec{a}(\vec{x}_f - \vec{x}_i)} = \sqrt{0^2 + 2(-9.81)(0 - 127.55)} = 50\ m/s$$

Assuming this experiment were performed in a vacuum (negating drag effects), we find that the final and initial speeds are equal, a result which agrees with conservation of energy.

Practise Exercise

A train travelling at a velocity $\vec{v}_i = 54\ km/h$ stops in $t_f = 2\,\text{min}$ after braking starts. Assuming the motion of the train during braking to be uniformly retarded, determine the distance covered during the braking time.

Example : (Projectile 2-dimensional motion).

Suppose that an object is not fired vertically but is fired at an angle θ from the ground. The object will then follow a parabolic trajectory, and its horizontal motion can be modeled independently of its vertical motion. Assume that the object is fired at an initial velocity of 50 m/s and 30 degrees from the horizontal. How far will it travel before hitting the ground?

The object experiences an acceleration of $-9.81\ m/s^2$ in the vertical direction and no acceleration in the horizontal direction. Therefore, the horizontal displacement is

$$\triangle\vec{x} = \vec{x}_f - \vec{x}_i = \vec{v}_i \cos\theta t + \frac{1}{2}\vec{a}t^2 = \vec{v}_i \cos\theta t$$

In other to solve this equation, we must find t. This can be done by analyzing the motion in the vertical direction. If we impose that the vertical displacement is zero, we can use the same procedure we did for rectilinear motion to find t.

$$0 = \vec{v}_i \sin\theta t + \frac{1}{2}\vec{a}t^2 = t(\vec{v}_i \sin\theta + \frac{1}{2}\vec{a}t)$$

We now solve for t and substitute this expression into the original expression for horizontal displacement.
(Note the use of the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$)

$$\triangle\vec{x} = \vec{v}_i \cos\theta \left(\frac{-2\vec{v}_i \sin\theta}{\vec{a}}\right) = -\frac{\vec{v}_i^2 \sin 2\theta}{\vec{a}} = 220.93\ m$$

Rotational motion

Rotational or angular kinematics is the description of the rotation of an object. The description of rotation requires some method for describing orientation, for example, the Euler angles. In what follows, attention is restricted to simple rotation about an axis of fixed orientation.
The z-axis has been chosen for convenience.
Description of rotation then involves these three quantities:
Angular position: The oriented distance from a selected origin on the rotational axis to a point of an object is a vector $\vec{r}(t)$ locating the point. The vector $\vec{r}(t)$ has some projection (or, equivalently, some component) $\vec{r}\perp(t)$ on a plane perpendicular to the axis of rotation. Then the angular position of that point is the angle θ from a reference axis (typically the positive x-axis) to the vector $\vec{r}\perp(t)$ in a known rotation sense (typically given by the right-hand rule).
Angular velocity: The angular velocity ω is the rate at which the angular position θ changes with respect to time t:

$$\omega = \frac{d\theta}{dt}$$

The angular velocity vector Ω points up for counterclockwise rotation and down for clockwise rotation, with magnitude ω and sense determined by the direction of rotation as given by the right-hand rule.
Angular acceleration: The magnitude of the angular acceleration α is the rate at which the angular velocity ω changes with respect to time t.

$$\alpha = \frac{d\omega}{dt}$$

The equations of translational kinematics can easily be extended to planar rotational kinematics with simple variable exchanges:

$$\begin{aligned}
&\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2 \\
&\theta_f - \theta_i = \frac{1}{2}(\omega_f + \omega_i)t \\
&\omega_f = \omega_i + \alpha t \\
&\alpha = \frac{\omega_f - \omega_i}{t} \\
&\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i).
\end{aligned}$$

Here θ_i and θ_f are, respectively, the initial and final angular positions, $\omega)i$ and ω_f are, respectively, the initial and final angular velocities, and α is the constant angular acceleration.

Although position in space and velocity in space are both true vectors (in terms of their properties under rotation), as is angular velocity, angle itself is not a true vector.

Point object in circular motion

This example deals with a "point" object, by which is meant that complications due to rotation of the body itself about its own center of mass are ignored.

Displacement: An object in circular motion is located at a position $\vec{r}(t)$ given by:

$$\vec{r}(t) = R\overrightarrow{U}_R(t),$$

where $\overrightarrow{U_R}$ is a unit vector pointing outward from the axis of rotation toward the periphery of the circle of motion, located at a radius R from the axis.

Linear velocity: The velocity of the object is then

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = R\frac{d}{dt}\overrightarrow{U}_R(t).$$

The magnitude of the unit vector $\overrightarrow{U}_R$ (by definition) is fixed, so its time dependence is entirely due to its rotation with the radius to the object, that is,

$$\frac{d}{dt}\overrightarrow{U}_R(t) = \overrightarrow{\Omega} \times \overrightarrow{U}_R = \omega(t)\overrightarrow{U}_\theta,$$

where $\overrightarrow{U}_\theta$ is a unit vector perpendicular to $\overrightarrow{U}_R$ pointing in the direction of rotation, $\omega(t)$ is the (possibly time varying) angular rate of rotation, and the symbol $\times$ denotes the vector cross product.
The velocity is then

$$\vec{v}(t) = R\omega(t)\overrightarrow{U}_\theta$$

The velocity therefore is tangential to the circular orbit of the object, pointing in the direction of rotation, and increasing in time if ω increases in time.

Linear acceleration: In the same manner, the acceleration of the object is defined as:

$$\begin{aligned}
\vec{a}(t) &= \frac{d}{dt}\vec{v}(t) = R\frac{d}{dt}\omega\overrightarrow{U}_\theta \\
&= \overrightarrow{U}_\theta R\frac{d}{dt}\omega + R\omega\frac{d}{dt}\overrightarrow{U}_\theta \\
&= \overrightarrow{U}_\theta R\frac{d}{dt}\omega + R\omega\overrightarrow{\Omega}\times\overrightarrow{U}_\theta \\
&= \overrightarrow{U}_\theta R\frac{d}{dt}\omega - \overrightarrow{U}_R\omega^2 R \\
&= \vec{a}_\theta + \vec{a}_R,
\end{aligned}$$

which shows a leading term $\vec{a}_\theta$ in the acceleration tangential to the orbit related to the angular acceleration of the object (supposing ω to vary in time) and a second term $\vec{a}_R$ directed inward from the object toward the center of rotation, called the centripetal acceleration.

Practise Exercise

The motion of a particle is described by the equations:

$$x = a\sin\omega t, \quad z = a\cos\omega t, \quad z = ut,$$

where a, ω and u are constants. Determine the path, velocity and acceleration of the particle.

Coordinate systems

In any given situation, the most useful coordinates may be determined by constraints on the motion, or by the geometrical nature of the force causing or affecting the motion. Thus, to describe the motion of a bead constrained to move along a circular hoop, the most useful coordinate may be its angle on the hoop. Similarly, to describe the motion of a particle acted upon by a central force, the most useful coordinates may be polar coordinates.

Fixed rectangular coordinates

In this coordinate system, vectors are expressed as an addition of vectors in the x, y, and z direction from a non-rotating origin. Usually, $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors in the $x-$, $y-$, and $z-$directions.

The position vector, $\vec{r}$ (or $\vec{s}$), the velocity vector, $\vec{v}$, and the acceleration vector, $\vec{a}$ are expressed using rectangular coordinates in the following way:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$
\begin{aligned}
\vec{v} &= \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} \\
\vec{a} &= \ddot{\vec{r}} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}
\end{aligned}
$$

Note: $\dot{x} = \dfrac{dx}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$.

Two dimensional rotating reference frame.

This coordinate system expresses only planar motion. It is based on three orthogonal unit vectors: the vector $\vec{i}$, and the vector $\vec{j}$ which form a basis for the plane in which the objects we are considering reside, and $\vec{k}$ about which rotation occurs.

Unlike rectangular coordinates, which are measured relative to an origin that is fixed and non-rotating, the original of these coordinates can rotate and translate - often following a particle on a body that is being studied.

Derivatives of unit vectors

The position, velocity, and acceleration vectors of a given point can be expressed using these coordinate systems, but we have to be a nit more careful than we do with fixed frames of reference. Since the frame of reference is rotating, the unit vectors also rotate, and this rotation must be taken into account when taking the derivative of any of these vectors.

If the coordinate frame is rotating at angular rate ω in the counterclockwise direction (that is, $\overrightarrow{\Omega} = \omega\vec{k}$ using the right hand rule) then the derivatives of the unit vectors are as follows:

$$
\begin{aligned}
\vec{i} = \omega\vec{k} \times \vec{i} = \omega\vec{j} \\
\vec{j} = \omega\vec{k} \times \vec{j} = \omega\vec{i}
\end{aligned}
$$

Position, velocity, and acceleration

Given these identities, we can now figure out how to represent the position, velocity, and acceleration vectors of a particle using this reference frame.

Position

Position is straightforward:

$$\vec{r} = x\vec{i} + y\vec{j}$$

It is just the distance form the origin in the direction of each of the unit vectors.

Velocity

Velocity is the time derivative of position:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(x\vec{i})}{dt} + \frac{d(y\vec{j})}{dt}$$

By the product rule, this is:

$$\vec{v} = \dot{x}\vec{i} + x\,\dot{\vec{i}} + \dot{y}\vec{j} + \dot{\vec{j}}$$

which from the identities above we know to be:

$$\vec{v} = \dot{x}\vec{i} + x\omega\vec{j} + \dot{y}\vec{j} - y\omega\vec{i} = (\dot{x} - y\omega)\vec{i} + (\dot{y} + x\omega)\vec{j}$$

or equivalently

$$\vec{v} = (\dot{x}\vec{i} + \dot{y}\vec{j}) + (y\,\dot{\vec{j}} + x\,\dot{\vec{i}}) = \vec{v}_{\text{rel}} + \overrightarrow{\Omega} \times \vec{r}$$

where $\vec{v}_{\text{rel}}$ is the velocity of the particle relative to the rotating coordinate system.

Acceleration

Acceleration is the time derivative of velocity.
We know that:

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d\vec{v}_{\text{rel}}}{dt} + \frac{d}{dt}\overrightarrow{\Omega} \times \vec{r}$$

Consider the $\frac{d}{dt}\vec{v}_{\text{rel}}$ part.
$\vec{v}_{\text{rel}}$ has two parts we want to find the derivative of: the relative change in velocity ($\vec{a}_{\text{rel}}$), and the change in the coordinate frame ($\overrightarrow{\Omega} \times \vec{v}_{\text{rel}}$).

$$\frac{d\vec{v}_{\text{rel}}}{dt} = \vec{a}_{\text{rel}} + \overrightarrow{\Omega} + \vec{v}_{\text{rel}}$$

Next, consider $\frac{d}{dt}(\overrightarrow{\Omega} \times \vec{r})$.
Using the chain rule:

$$\frac{d(\overrightarrow{\Omega} \times \vec{r})}{dt} = \dot{\overrightarrow{\Omega}} \times \vec{r} + \overrightarrow{\Omega} \times \dot{\vec{r}}$$

$\dot{\vec{r}} = \vec{v} = \vec{v}_{\text{rel}} + \overrightarrow{\Omega} + \vec{r}$ from above:

$$\frac{d(\overrightarrow{\Omega} \times \vec{r})}{dt} = \dot{\overrightarrow{\Omega}} \times \vec{r} + \overrightarrow{\Omega} \times (\overrightarrow{\Omega} \times \vec{r}) + \overrightarrow{\Omega} \times \vec{v}_{\text{rel}}$$

So all together:

$$\vec{a} = \vec{a}_{\text{rel}} + \overrightarrow{\Omega} \times \vec{v}_{\text{rel}} + \dot{\overrightarrow{\Omega}} \times \vec{r} + \overrightarrow{\Omega} \times (\overrightarrow{\Omega} \times \vec{r}) + \overrightarrow{\Omega} \times \vec{v}_{\text{rel}}$$

And collecting terms:

$$\vec{a} = \vec{a}_{\text{rel}} + 2(\overrightarrow{\Omega} \times \vec{v}_{\text{rel}}) + \dot{\overrightarrow{\Omega}} \times \vec{r} + \overrightarrow{\Omega} \times (\overrightarrow{\Omega} \times \vec{r}).$$

Kinematic constraints

A kinematic constraint is any condition relating properties of a dynamic system that must hold true at all times. Below are some common examples:

Rolling without slipping

An object that rolls against a surface without slipping obeys the condition that the velocity of its center of mass is equals to the cross product of its angular velocity with a vector point of contact to the centre of mass,

$$\overrightarrow{V}_G(t) = \overrightarrow{\Omega} \times \vec{r}_{G/0}$$

For the case of an object that does not tip or turn, this reduces to $\overrightarrow{V} = R\omega$.

Inextensible cord

This is the case where bodies are connected by some cord that remains in tension and cannot change length. The constraint is that the sum of all components of the cord is the total length, and accordingly the time derivative of this sum is zero. A dynamic problem of this type is the pendulum.

Another example is a drum turned by the pull of gravity upon a falling weight attached to the rim by the inextensible cord.

Summary

Let $\vec{r}$ be the position vector of a moving particle, an instantaneous velocity is defined as:
$\vec{v} = \dfrac{d\vec{r}}{dt}$,
average velocity is defined as
$\vec{v}_{\text{ave}} = \dfrac{\triangle \vec{r}}{\triangle t}$
where $\vec{v}_{\text{ave}} = \vec{v} = \lim\limits_{\triangle t \to 0} \dfrac{\triangle \vec{r}}{\triangle t}$;
and instantaneous acceleration is defined as:
$\vec{a} = \dfrac{d\vec{v}}{dt}$
average acceleration is defined as:

$\vec{v}_{\text{ave}} = \dfrac{\triangle\vec{v}}{\triangle t}$
where
$\vec{a}_{\text{ave}} = \vec{a} = \lim\limits_{\triangle t\to 0} \dfrac{\triangle\vec{v}}{\triangle t}.$

Types of motion
Uniform motion implies constant velocity in a straight line.
Non-uniform motion implies acceleration.
Integral relations of the above formula are given as follows:

$\vec{v}(t) = \vec{v}_0 + \displaystyle\int_0^t dt'\vec{a}(t')$

$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \displaystyle\int_0^t dt'(t-t')\vec{a}(t')$, For constant acceleration

$\vec{v} = \displaystyle\int_0^t \vec{a}\ dt' = \vec{v}_0 + \vec{a}t,$

$\vec{r} = \vec{r}_0 + \displaystyle\int_0^t \vec{v}\ dt' = \vec{r}_0 + \int_0^t (\vec{v}_0 + \vec{a}t)dt'$

$= \vec{r}_0 + \vec{v}_0 t + \dfrac{1}{2}\vec{a}t^2$

Substituting for $\vec{v}$, we get

$\vec{r} = \vec{r}_0 + \dfrac{\vec{v}+\vec{v}_0}{2}t.$

For the case where $\vec{r}$ is parallel to $\vec{a}$ resulting in a straight-line motion, the vector $\vec{r}$ has magnitude equal to the path length s at time t,

$v^2 = v_0^2 + 2as,$

with reference objects, relative velocity and acceleration are given, respectively as

$\overrightarrow{V}_{A/B} = \overrightarrow{V}_A - \overrightarrow{V}_B,$

and

$\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B$

Rotational or angular kinematics is the description of the rotation of an object.
Description of rotation involves three quantities:
Angular position of a point is the angle θ from a reference axis to the vector $\vec{r}\perp(t)$ in a known rotation sense.

Angular velocity

$$\omega = \frac{d\theta}{dt}$$

and

Angular acceleration

$$\alpha = \frac{d\omega}{dt}.$$

The equations of translational kinematics can easily be extended to planar rotational kinematics. With simple variable exchanges

$$\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\theta_f - \theta_i = \frac{1}{2}(\omega_f + \omega_i)t.$$

For an object in circular motion, its displacement is given as:

$$\vec{r}(t) = R\overrightarrow{U}_R(t),$$

its linear velocity is given as

$$\overrightarrow{V}(t) = \frac{d}{dt}\vec{r}(t) = R\frac{d}{dt}\overrightarrow{U}_R(t)$$

where

$$\frac{d}{dt}\overrightarrow{U}_R(t) = \overrightarrow{\Omega} \times \overrightarrow{U}_R = \omega(t)\overrightarrow{U}_\theta$$

so that we have

$$\overrightarrow{V}(t) = R\omega(t)\overrightarrow{u}_\theta$$

and its linear acceleration is given as:

$$\vec{a}(t) = R\frac{d}{dt}\omega\vec{u}_\theta - R\omega^2\overrightarrow{U}_R$$

$$= \vec{a}_\theta + \vec{a}_R.$$

Position vector, velocity, and acceleration are also represented in coordinates system as:

For fixed rectangular coordinates

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v} = \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

$$\vec{a} = \ddot{\vec{r}} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

where

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}$$

The above coordinates system in two dimensional rotating reference frame with derivatives of unit vectors lead to obtaining:

$$\overrightarrow{V} = \dot{x}\vec{i} + x\,\dot{\vec{i}} + \dot{y}\vec{j} + y\,\dot{\vec{j}}$$
$$= \dot{x}\vec{i} + x\omega\vec{j} + \dot{y}\vec{j} - y\omega\vec{i}$$
$$= (\dot{x} - y\omega)\vec{i} + (\dot{y} + x\omega)\vec{j}$$
$$= \vec{v}_{\text{rel}} + \Omega \times \vec{r}$$
$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d\vec{v}_{\text{rel}}}{dt} + \frac{d}{dt}\overrightarrow{\Omega} \times \vec{r}$$
$$\vec{a} = \vec{a}_{\text{rel}} + 2(\overrightarrow{\Omega} \times \vec{v}_{\text{rel}}) + \dot{\overrightarrow{\Omega}} \times \vec{r} + \overrightarrow{\Omega} \times (\overrightarrow{\Omega} \times \vec{r}),$$

and finally a kinematic constraint is any condition relating properties of a dynamic system that must hold true at all times; for example, an object that rolls against a surface without slipping, another example is a drum turned by the pull of gravity upon a falling weight attached to the rim by the inextensible cord.

Post-Test

1. A man of height h walks away from a lamp hanging at a height H with a velocity $\vec{u}$. Determine the velocity of the tip of the man's shadow.

2. The motion of a particle is described by the equations

$$x = 8t - 4t^2, \quad y = 6t - 3t^2,$$

where x and y are in metres and t is in seconds.
Determine the path, velocity and acceleration of the particle.

3. The motion of a particle is described by the equations:

$$x = a\sin\omega t, \quad y = a\cos\omega t, \quad z = ut,$$

where a, ω and u are constants. Determine the path, velocity and acceleration of the particle.

4. A train starts moving from rest with uniform acceleration along a curve of radius $R = 800\ m$ and reaches a velocity $v_f = 36\ km/h$ after travelling a distance $s_f = 600\ m$. Determine the velocity and acceleration of the train at the middle of this distance.

5. A flywheel of radius $R = 1.2\ m$ rotates uniformly, making $n = 90\ rpm$. Determine the linear velocity and acceleration of a point on the rim of the flywheel.

6. The center O of a wheel of radius $R = 0.2\ m$ rolling along a straight rail has at a given instant a velocity $\vec{v}_0 = 1\ m/sec$ and an acceleration $\vec{v}_0 = 2\ m/sec^2$. Determine the acceleration of point B lying at the end of diameter AB perpendicular to OP and the acceleration of point P coincident with the instantaneous centre of zero velocity.

7. The current of a river of width h has a constant velocity $\vec{v}$. A man can row a boat in motionless water with a velocity $\vec{u}$. Determine the direction he should take in order to cross the river in the least possible time and the point where he will reach the opposite bank.

Supplementary Reading

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2. L.J. Meriam and L.G. Kraige, Engineering Mechanics: Statics and Dynamics, 5th edition, John Wiley & Sons, Inc., New York, NY, 2002.

3. F.P. Beer and E.R. Johnston, Vector Mechanics for Engineers (Statics). McGraw-Hill, 3rd Edition (1999).

4. Bedford/Fowler, Statics and Dynamics, Engineering Mechanics, 3rd edition, Prentice Hall, Upper Saddle River, New Jersey, 2002.

LECTURE 8

Linear Impulse and Momentum

Introduction

In this lecture we will consider the equations that result from integrating Newton's Second Law, $F = ma$, in time. This will lead to the principle of linear impulse and momentum. This principle is very useful when solving problems in which we are interested in determining the global effect of a force acting on a particle over a time interval.

Objectives

At the end of this lecture you should be able to:

- analyze the motion of an object,
- understand the relation between Newton's Second Law and linear momentum,
- develop the principle of linear impulse and momentum,
- apply the conservation of linear momentum principle to derive the equation of motion, and
- further develop your ability to define and solve problems in linear impulse and momentum.

Pre-Test: (See Post-Test)

Newton's Second Law

We consider the curvilinear motion of a particle of mass, m, under the influence of a force $\overrightarrow{F}$.

Assuming that the mass does not change, Newton's Second Law states

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v})$$

Linear Momentum

The linear momentum vector, $\vec{P}$, is defined as

$$\vec{P} = m\vec{v}$$

Thus, an alternative form of Newton's Second Law is

$$\vec{F} = \dot{\vec{P}} = \frac{d}{dt}\vec{P}, \tag{1}$$

which states that the total force acting on a particle is equal to the time rate of change of its linear momentum.

Example 1

A model airplane is 1 Kg travelling due north at 1 m/s in straight and level flight has a momentum of 1 $Kg\ m/s$ due north measured from the ground. To the dummy pilot is the Cockpit it has a velocity and momentum of zero.

Example 2

A model airplane of 1 Kg accelerates from rest to a velocity of 1 m/s due north in 1 s. The thrust required to produce this acceleration is 1 newton.

Note

It is obvious that linear impulse and momentum have the same units. In the SI system they are $N \cdot s$ or $Kg \cdot m/s$, whereas in English system they are $Ib \cdot s$, or slug. ft/s.

Linear momentum of a system of particles

The linear momentum of a system of particles is the vector sum of the momenta of all the individual objects in the system:

$$\vec{P} = \sum_{i=1}^{n} m_i\vec{v}_i = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{n}$$

where $\vec{P}$ is the total momentum of the particle system, m_i and $\vec{v}_i$ are the mass and the velocity vector of the i-th object, and n is the number of objects in the system.

Principle of Linear Impulse and Momentum

Imagine now that the force considered acts on the particle between time t_1 and time t_2.

Equation (1) can then be integrated in time to obtain

$$\int_{t_1}^{t_2} \vec{F}dt = \int_{t_1}^{t_2} \dot{\vec{P}}\, dt = \vec{P}_2 - \vec{P}_1 = \triangle\vec{P} \tag{2}$$

Here, $\overrightarrow{P}_1 = \overrightarrow{P}(t_1)$ and $\overrightarrow{P}_2 = \overrightarrow{P}(t_2)$.
The term

$$\overrightarrow{I} = \int_{t_1}^{t_2} \overrightarrow{F} dt$$

is called the linear impulse. Thus, the linear impulse on a particle is equal to the linear momentum change.

Example 3

A load of weight $P = 0.1\ Kg$ moves in a circle with a constant velocity $\vec{v} = 2\ m/s$. Determine the impulse during the time the load takes to travel one quarter of the circle.

Solution

From equation (2), the change in momentum,

$$\overrightarrow{I} = m\vec{v}_1 - m\vec{v}_0$$

Interpreting geometrically, consider the Figure 30 below

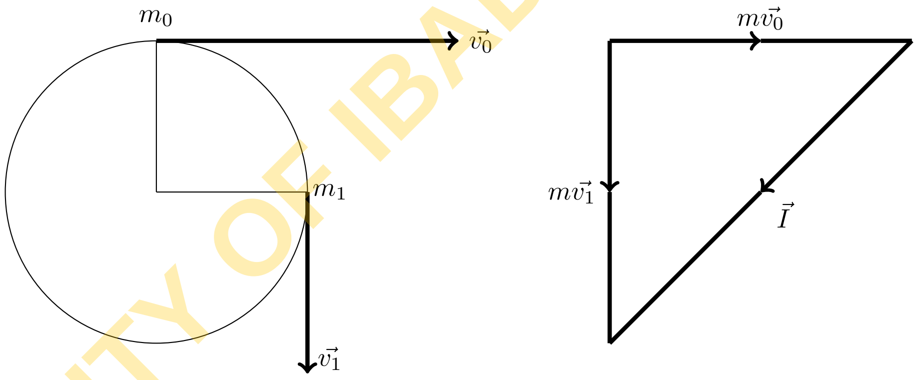


Figure 30:

We find from the right-angled triangle:

$$|\overrightarrow{I}| = m\sqrt{|\vec{v}_1^2| + |\vec{v}_0^2|}$$

But from the condition of the problem $|\vec{v}_1| = |\vec{v}_0| = |\vec{v}|$, consequently,

$$\overrightarrow{I} = \frac{P}{a}\vec{v}\sqrt{2} = 0.029\ Kg\ sec$$

Note

It is obvious that linear impulse and momentum have the same units. In the SI system they are $N \cdot s$ or $Kg.m/s$, whereas in English system they are $Ib.s$, or slug.ft/s.

Impulsive Forces

We typically think of impulsive forces as being forces of very large magnitude that act over a very small interval of time, but cause a significant change in the momentum. Examples of impulsive forces are those generated when a ball is hit by a tennis racquet or a baseball bat, or when a steel ball bounces on a steel plate.

Conservation of Linear Momentum

We see from equation (1) that if the resultant force on a particle is zero during an interval of time, then its linear momentum $\overrightarrow{P}$ must remain constant. Since equation (1) is a vector quantity, we can have situations in which only some components of the resultant force are zero. For instance, in Cartesian coordinates, if the resultant force has a non-zero component in the y direction only, then the x and z components of the linear momentum will be conserved since the force components in x and z are zero.

Consider now two particles, m_a and m_b, which interact during an interval of time. Assume that interaction forces between them are the only unbalanced forces on the particles. Let $\overrightarrow{F}$ be the interaction force that particles m_b exerts on particle m_a. Then, according to Newton's third law, the interaction force that particle m_a exerts on particle m_b will be $-\overrightarrow{F}$.

Using expression (2), we will have that $\triangle\overrightarrow{P}_a = -\triangle\overrightarrow{P}_a$ or $\triangle\overrightarrow{P} = \triangle\overrightarrow{P}_a + \triangle\overrightarrow{P}_b = \vec{0}$. That is, the change of momentum of particles m_a and m_b are equal in magnitude and opposite in sign, and the total momentum change equals zero. Recall that this is true if the only unbalanced forces on the particles are the interaction force.

Remark

A common problem in physics that requires the use of this fact is the collision of two particles. Since momentum is always conserved, the sum of the momenta before the collision must equal the sum of the momenta after the collision:

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

where
$\vec{u}$ signifies vector velocity before the collision,
$\vec{v}$ signifies vector velocity after the collision.
Correctly solving this problem means you have to know what kind of collision took place.

There are two basic kinds of collisions, both of which conserve momentum:

- Elastic collisions conserve kinetic energy as well as total momentum before and after collision.

- Inelastic collisions don't conserve kinetic energy, but total momentum before and after collision is conserved.

Elastic collisions

A collision between two pool balls is a good example of an almost totally elastic collision; a totally elastic collision exists only in theory.
In addition to momentum being conserved when the two balls collide, the sum of kinetic energy before a collision must equal the sum of kinetic energy after

$$\frac{1}{2}m_1\vec{v}_{1,i}^2 + \frac{1}{2}m_2\vec{v}_{2,i}^2 = \frac{1}{2}m_2\vec{v}_{1,f}^2 + \frac{1}{2}m_2\vec{v}_{2,f}^2$$

$\frac{1}{2}$ factor is common, so it can be taken out right away.

Head-on collision (1 dimensional)

In the case of two objects colliding head on we find that the final velocity

$$\begin{aligned}\vec{V}_{1,f} &= \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{V}_{1,i} + \left(\frac{2m_2}{m_1 + m_2}\right)\vec{V}_{2,i}\\ \vec{V}_{2,f} &= \left(\frac{2m_1}{m_1 + m_2}\right)\vec{V}_{1,i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\vec{V}_{2,i}\end{aligned}$$

Using the following approximations:
The case when the mass of one body, say m_1, is far greater than that of the other, m_2, this means $m_1 + m_2 \to m_1$, $m_1 - m_2 \to m_1$ and the above formula for $\vec{V}_{2,f}$ reduces to

$$\vec{V}_{2,f} = 2\vec{V}_{1,i} - \vec{V}_{2,i}$$

Remark

Its physical interpretation is that in the case of collision between two bodies, one of which is much more massive than the other, the lighter body ends up moving in the opposite direction with twice the original speed of the more massive body.

Another special case is when the collision is between two bodies of equal mass. Say body m_1 moving at velocity $\vec{v}_1$ strikes body m_2. Putting this case in the equation derived above, we will see that after the collision, the body that has moving (m_1) will start moving with velocity $\vec{v}_2$ and the mass m_2 will start moving with velocity $\vec{v}_1$.

Remark

Now suppose one of the masses, say m_2, was at rest. In that case after the collision the moving body, m_1 will come to rest and the body that was at rest, m_2, will start moving with the velocity that m_1 had before the collision.

Multi-dimensional collisions

In the case of objects colliding in more than one dimension, as in oblique collisions, the velocity is resolved into orthogonal components with one component perpendicular to the plane of collision and the other component or components in the plane of collision.

Remark

The velocity components in the plane of collision remain unchanged, while the velocity perpendicular to the plane of collision is calculated in the same way as the one-dimensional case.

Inelastic collisions

A common example of a perfectly inelastic collision is when two snowballs collide and the stick together afterwards. This equation describes the conservation of momentum:

$$m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i} = (m_1 + m_2)\vec{v}_f$$

Remark

It can be shown that a perfectly inelastic collision is one in which the maximum amount of kinetic energy is converted into other forms. When a collision is not elastic it is an inelastic collision.

In case of a perfectly inelastic collision the relation velocity of separation of the centre of masses of the colliding bodies is $\vec{0}$. Hence after collision the bodies stick together.

In all types of collision if no external force is acting on the system of colliding bodies, the momentum will always be preserved.

Explosions

An explosion occurs when an object is divided into two or more fragments due to a release of energy.

Note that kinetic energy in a system of explosion is not conserved because it involves energy transformation (i.e., kinetic energy changes into heat and sound energy).

Summary

The equation which govern the motion of a single particle is

$$\overrightarrow{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

where $\vec{r}$ is the vector displacement of the particle with respect to a fixed origin, $\overrightarrow{F}$ is the resultant force acting on the particle.

Suppose we now apply these results for a single particle to the motion of a system of mutually interacting particles.

Let there be n particles in the system, and let their masses be denoted by $m_1, m_2, \ldots, m_n$, then the linear momentum vector $\overrightarrow{P}$, is defined as

$$\overrightarrow{P} = \sum_{i=1}^{n} m_i \vec{v}_i. \tag{3}$$

The term

$$I = \int_{t_1}^{t_2} F\,dt$$

is called the linear impulse on a particle and is equal to the linear momentum change. The law of conservation of linear momentum

(1) Let the sum of all the external forces acting on a system be zero:

$$\sum F = 0$$

In this case, Equation (3) is constant.

Thus, if the sum of all the external forces acting on a system is zero, the momentum vector, $\overrightarrow{P}$, of the system is constant in magnitude and direction.

(2) If the sum of the projections on any axis of all the external forces acting on a system is zero, the projection of the momentum of that system on that axis is a constant quantity.

It follows from the above that internal forces are incapable of changing the total momentum of a system.

Post-Test

1. A car of mass m_1 travelling at velocity v passes a car of mass m_2 parked at the side of the road. The momentum of the system of two cars is frictionless surface as shown
2. see Figure 31 below

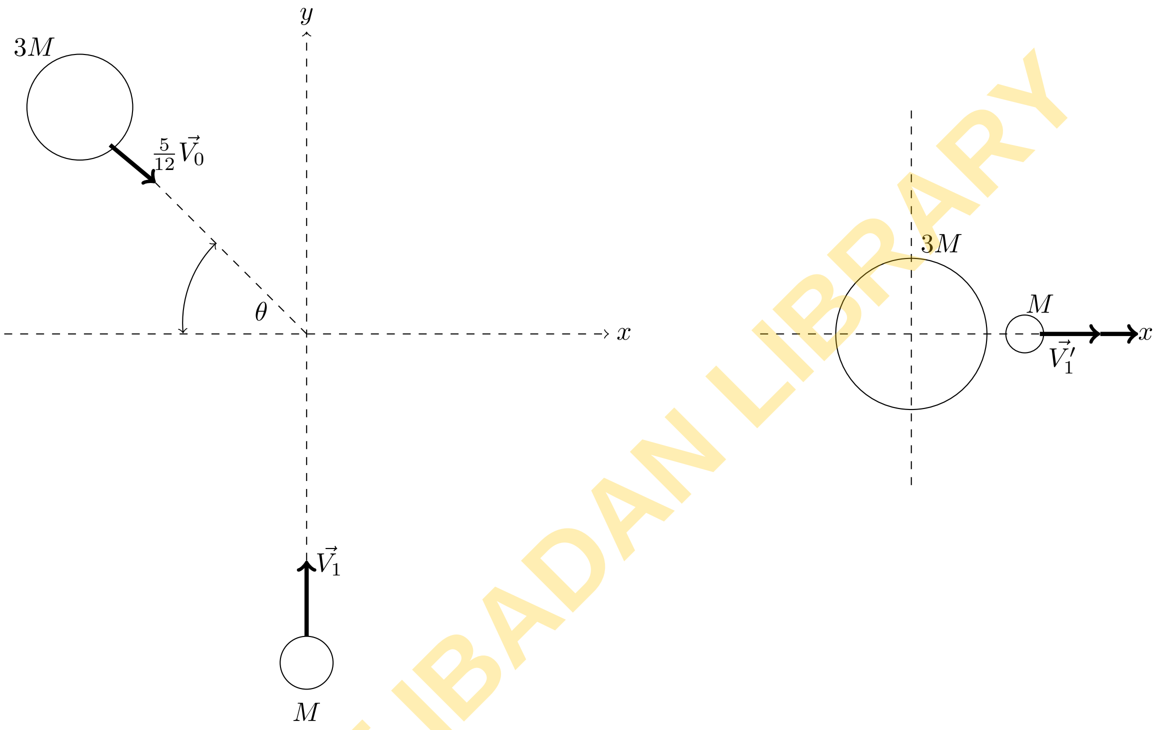


Figure 31:

Before **After**

Two masses M and $3M$ collide on a horizontal frictionless surface as shown.(See Fig. 31) Before the collision, the mass M has a velocity $\overrightarrow{V}_1$ in the y-direction. The mass $3M$ has a velocity $\dfrac{5}{12}\overrightarrow{V}_0$ making an angle θ to the x-axis as shown. Assume $\sin\theta = 3/5$ and $\cos\theta = 4/5$. After the collision the mass $3M$ is at rest. The mass M moves along the x-axis with the velocity $\overrightarrow{V}'_1$. Neglect gravity.
Give all your answers to parts (b), (c) and (d) in terms of M and $\overrightarrow{V}_0$. Be careful do not confuse your symbols.

(a) What are the x and y-components of the net linear momentum before the collision in terms of M, $\overrightarrow{V}_1$, $\overrightarrow{V}_0$ and θ?

(b) What is the speed V_1 of the mass M before the collision?

(c) What is the speed V_1' of the mass M after the collision?

(d) What is the velocity of the center-of-mass?

3. A bullet of mass m_1 is fired into a pendulum of mass m_2 and length L. The speed of the bullet as it enters the mass m_2 is V_1 (see Figure 32 below).

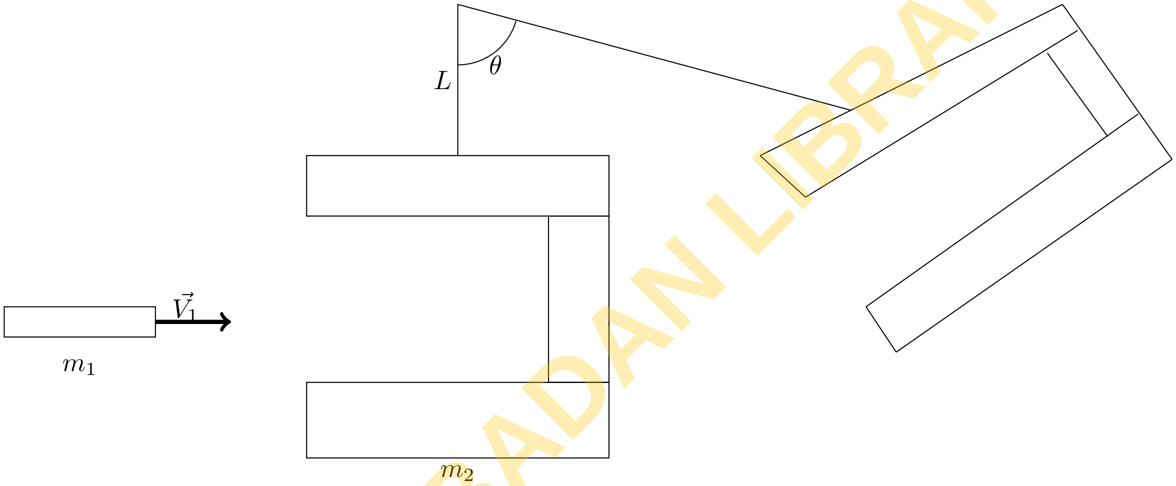


Figure 32:

First, assume that the collision is elastic, and that $m_1 \ll m_2$.

(a) If the pendulum is initially at rest, what is the speed of the bullet after the collision?

(b) Now suppose that when the collision occurs, the pendulum, at the bottom of its swing, is moving to the lest with velocity V_2. What now is the speed of the bullet after the elastic collision?
Now assume that the collision is completely inelastic. The pendulum is at rest before the collision $m_1 < m_2$, but the speed V_1 of the bullet is unknown.

(c) After the collision the pendulum moves to the right and it comes to a half when the string makes an angle $\theta_{\max}$ with the vertical. What was the speed of the bullet? Substitute in your answer $\theta_{\max} = 0$. Does your result make sense?

(d) Could $\theta_{\max}$ be 90^0. Explain your answer.

4. A particle is moving in three dimensions. Its position vector is given by:

$$\vec{r} = 6\hat{x} + (3 + 4t)\hat{y} - (3 + 2t - t^2)\hat{z}$$

Distances are in meters, and the time t, in seconds.

(a) What is the velocity vector at $t = +3$?

(b) What is the speed (in m/sec) at $t = +3$?

(c) What is the acceleration vector and what is its magnitude (in $m/\sec^2$) at $t = +3$?

Now the particle is moving only along the z-axis, and its position is given by

$$(t^2 - 2t - 3)\hat{z}.$$

(d) At what time does the particle stand still?

(e) Make a plot (a sketch) of z versus time covering $t = -2$ to $+4$ sec.

5. A particle of mass m_1 and speed v_1 (in the $+x$-dimensional) collides with another particle of mass m_2. m_2 is at rest before the collision occurs, thus $v_2 = 0$. After the collision, the particles have velocities v_1' and v_2' in the $x - y$ plane in the direction of θ_1 and θ_2 with the x-axis. There are no external forces. Express all your answers in terms of m_1, m_2, v_1, θ_1 and θ_2.

   (a) What is the total momentum before the collision (direction and magnitude)?

   (b) What is the total momentum after the collision (direction and magnitude)?

Supplementary Reading

1. Serway, Raymond; Jewett, John (2004). Physics for Scientists and Engineers (6th ed.). Brooks/Cole. ISBN 0-534-40842-7.

2. Tipler, P. (2004). Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

3. Hand, Louis N.; Finch, Janet D. Analytical Mechanics. Cambridge University Press. Chapter 4.

4. Lanczos, Cornelius (1970). The variational Principles of Mechanics. Toronto: University of Toronto Press. ISBN 0-8020-1743-6.

LECTURE 9

Angular Momentum and its Conservation

Introduction

The angular momentum of a particle about an origin is a vector quantity related to rotation, equal to the mass of the particle multiplied by the cross product of the position vector of the particle with its velocity vector.

The angular momentum of a system of particle is the sum of that of the particles within it. Angular momentum is an important concept with numerous applications. The conservation of angular momentum explains many phenomena in nature.

Objective

At the end of this lecture you should be able to use the vector product and the right-hand rule, so you can:

- Calculate the torque of a specified force about an arbitrary origin.
- Calculate the angular momentum vector for a moving particle.

You should understand angular momentum conservation, so you can:

- Recognize the conditions under which the law of conservation is applicable and relate this law to one- and two-particle systems such as satellite orbits.
- State the relation between net external torque and angular momentum, and identify situations in which angular momentum is conserved.
- Analyze problems in which the moment of inertia of an object is changed as it rotates freely about a fixed axis.

Pre-Test (See Post-Test)

Angular Momentum in Classical Mechanics

Definition:

Angular momentum of a particle about a given origin is defined as:

$$\overrightarrow{L} = \vec{r} \times \vec{p}$$

where
$\overrightarrow{L}$ is the angular momentum of the particle,
$\vec{r}$ is the position vector of the particle relative to the origin.
$\vec{p}$ is the linear momentum of the particle, and X is the vector cross product.

As seen from the definition, the derived SI units of angular momentum are Newton metre seconds (N.M.S. or $kg.m^2.s^{-1}$) or joule seconds. Because of the cross product, $\overrightarrow{L}$ is a pseudovector perpendicular to both the radial vector $\vec{r}$ and the momentum vector $\vec{p}$ and it is assigned a sign by the right-hand rule.

Angular Momentum of a Collection of Particles

If a system consists of several particles, the total angular momentum about an origin can be obtained by adding (or integrating) all the angular momenta of the constituent particles. Angular momentum can also be calculated by multiplying the square of the displacement $\vec{r}$, the mass of the particle and the angular velocity.

Angular Momentum in the Centre of Mass Frame

It is very often convenient to consider the angular momentum of a collection of particles about their center of mass, since this simplified the mathematics considerably.

The angular momentum of a collection of particles is the sum of the angular momentum of each particle:

$$\overrightarrow{L} = \sum_i R_i \times m_i \overrightarrow{V}_i$$

where R_i is the distance of particle i from the reference point, m_i is its mass, and $\overrightarrow{V}_i$ is its velocity. The center of mass is defined by:

$$R = \frac{1}{M} \sum_i m_i R_i$$

where the total mass of all particles is given by:

$$M = \sum_i m_i$$

It follows that the velocity of the center of mass is

$$\overrightarrow{V} = \frac{1}{M} \sum_i m_i \overrightarrow{V}_i$$

If we define $\vec{r}_i$ as the displacement of particle i from the center of mass, and $\overrightarrow{V}_i$ as the velocity of particle i with respect to the center of mass, then we have

$$R_i = R + \vec{r}_i \quad \text{and} \quad \overrightarrow{V}_i = \overrightarrow{V} + \vec{v}_i$$

and also

$$\sum_i m_i\vec{r}_i = 0 \quad \text{and} \quad \sum_i m_i\vec{v}_i = 0$$

so that the total angular momentum is

$$\begin{aligned}\overrightarrow{L} &= \sum_i (R + r_i) \times m_i(\overrightarrow{V} + \vec{v}_i) \\ &= (R \times M\overrightarrow{V}) + \left(\sum_i \vec{r}_i \times m_i\vec{v}_i\right)\end{aligned}$$

Remark

The first term is just the angular momentum of the center of mass. It is the same angular momentum one would obtain if there were just one particle of mass M moving at velocity $\overrightarrow{V}$ located at the center of mass. The second term is the angular momentum that is the result of the particles moving relative to their center of mass. This second term can be even further simplified if the particles form a rigid body, in which case, a spin appears.

Fixed axis of rotation

For many applications where one is only concerned about rotation around one axis, it is sufficient to discard the pseudovector nature of angular momentum, and treat it like a scalar where it is positive when it corresponds to a counter-clockwise rotation, and negative clockwise.

To do this, just take the definition of the cross product and discard the unit vector, so that angular momentum becomes:

$$\overrightarrow{L} = |\vec{r}||\vec{p}| \sin\theta_{\vec{r},\vec{p}}$$

where $\theta_{\vec{r},\vec{p}}$ is the angle between $\vec{r}$ and $\vec{p}$ measured from $\vec{r}$ to $\vec{p}$; an important distinction because without it, the sign of the cross product would be meaningless. From the above, it is possible to reformulate the definition to either of the following:

$$\overrightarrow{L} = \pm|\vec{p}||\vec{r}_\perp|$$

where $\vec{r}_\perp$ is called the lever arm distance to $\vec{p}$. The easiest way to conceptualize this is to consider the lever arm distance to the distance from the origin to the line that $\vec{p}$ travels along.

With this definition, it is necessary to consider the direction of $\vec{p}$ (pointed clockwise or counter-clockwise) to figure out the sign of $\overrightarrow{L}$.

Equivalently,

$$\overrightarrow{L} = \pm|\vec{r}||\vec{p}_\perp|$$

where $\vec{p}_\perp$ is the component of $\vec{p}$ that is perpendicular to $\vec{r}$. As above, the sign is decided based on the sense of rotation.

For an object with a fixed mass that is rotating about a fixed symmetry axis, the angular momentum is expressed as the product of the moment of inertia of the object and its angular velocity vector.

$$\overrightarrow{L} = \overrightarrow{I}\omega$$

where $\overrightarrow{I}$ is the moment of inertia of the object (in general, a tensor quantity) ω is the angular velocity. As the kinetic energy K of a massive rotating body is given by

$$K = I\omega^2/2$$

it is proportional to the square of the angular momentum.

Conservation of Angular Momentum

In a closed system angular momentum is constant. This conservation law mathematically follows from continuous directional symmetry of space (no direction in space is any different from any other direction).

The time derivative of angular momentum is called torque:

$$\begin{aligned} r = \frac{d\overrightarrow{L}}{dt} &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\ &= 0 + \vec{r} \times \overrightarrow{F} \\ &= \vec{r} \times \overrightarrow{F} \end{aligned}$$

(The cross-product of velocity and momentum is zero, because these vectors are parallel.)

Summary

The angular momentum $\overrightarrow{L}$ for a system of particles with linear momenta $\vec{p}_j$ and distances $\vec{r}_j$ from the rotation axis is defined

$$\overrightarrow{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i = \sum_{i=1}^{r} m_i \vec{r}_i \times \vec{v}_i.$$

For a rigid body rotating with angular velocity ω about the rotation axis $\hat{n}$ (a unit vector), the velocity vector $\vec{v}_i$ may be written as a vector cross product

$$\vec{v}_i = \omega\hat{n} \times \vec{r}_i \overset{\text{def}}{=} \vec{\omega} \times \vec{r}_i$$

where angular velocity vector

$\vec{\omega} \overset{\text{def}}{=} \omega$ is r_i is the shortest vector from the rotation axis to the point mass. Substituting the formular form $\vec{v}_i$ into the definition of $\overrightarrow{L}$ yields

$$\overrightarrow{L} = \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \vec{\omega} \sum_{i=1}^{n} m_i r_i^2 = \overrightarrow{I}\omega\hat{n}$$

where we have introduced the special case that the position vectors of all particles are perpendicular to the rotation axis (e.g., a flywheel): $\vec{\omega} \cdot \vec{r}_i = \vec{0}$. The torque is defined as the rat of change of the angular momentum $\overrightarrow{L}$

$$\overrightarrow{N} \overset{\text{def}}{=} \frac{d\overrightarrow{L}}{dt}$$

If I is constant (because the inertia tensor is the identity, because we work in the intrinsecal frame, or because the torque is deriving the rotation around the same axis $\hat{n}$ so that I is not changing) then we may write

$$\overrightarrow{N} \overset{\text{def}}{=} \overrightarrow{I}\frac{d\omega}{dt}\hat{n} = \overrightarrow{I} \times \hat{n}$$

where α is called the angular acceleration (or rotational acceleration) about the rotation axis $\hat{n}$.

Notice that if I is not constant in the external reference frame (i.e., the three main axes of the body are different) then we cannot take I outside the derivate. In this cases, we can have torque-free precession.

Post-Test

1. A uniform cylindrical shell (hoop) sits on one of its flat sides on a frictionless surface. The hoop has mass M, radius R and height H. A bullet of mass μ moving horizontally with velocity $\vec{v}_0$ strikes the hoop with impact parameter R at mid-height ($H/2$ from the surface). After the collision the bullet continues with velocity $\vec{v}_0/2$ in its original direction. Ignore any hole the bullet creates. Give all your answers in terms of M, R, H, $\overrightarrow{V}_0$ and μ.
    (a) What was the angular momentum, $\overrightarrow{L}$, of the system about the center of the hoop before the collision?
    (b) What is the linear velocity, $\overrightarrow{V}$, of the center of the hoop after the collision?
    (c) What is the angular velocity, $\vec{\omega}$, of the hoop after the collision?
2. Two forces of magnitude 50 N , as shown in the figure below, act on a cylinder of radius 4 m and mass 6.25 Kg. The cylinder, which is initially at rest, sits on a frictionless surface. After 1 second, the velocity and angular velocity of the cylinder in m/s and rad/s are respectively (see Figure 33 below).

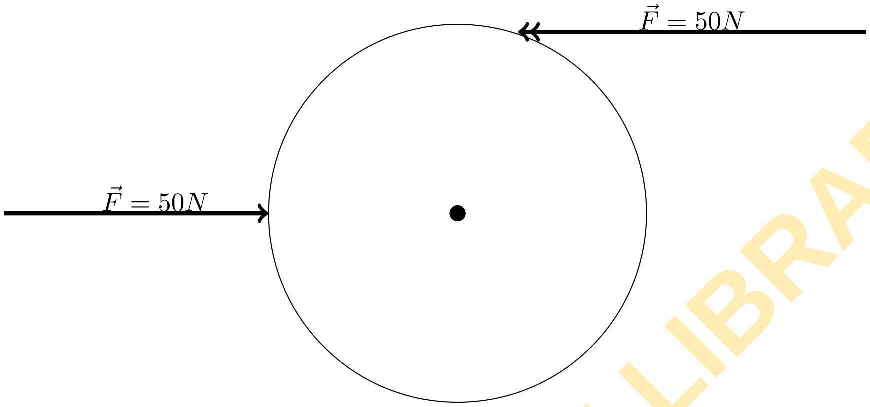


Figure 33:

Supplementary Reading

1. Serway, Raymond; Jewett, John (2004). Physics for Scientists and Engineers (6th ed.). Brooks/Cole. ISBN 0-534-40842-7.

2. Tipler, P. (2004). Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

3. Hand, Louis N.; Finch, Janet D. Analytical Mechanics. Cambridge University Press. Chapter 4.

4. Lanczos, Cornelius (1970). The variational Principles of Mechanics. Toronto: University of Toronto Press. ISBN 0-8020-1743-6.

LECTURE 10

Center of Mass

Introduction

The center of mass of a system of particles is a specific point at which, for many purposes, the system's mass behaves as if it were concentrated. The center of mass is a function only of the positions and masses of the particles that comprise the system.

In this lecture, we shall discuss how to locate and use integration to find center of mass of various objects.

Objective

At the end of this lecture, you should be able to:

- identify by inspection the center of mass of a symmetrical object,
- locate the center of mass of a system consisting of two such objects,
- use integration to find the center of mass of a thin rod of non-uniform density,
- understand and apply the relation between center-of-mass velocity and linear momentum, and between center-of-mass acceleration and net external force for a system of particles,
- define center of gravity and to use this concept to express the gravitational potential energy of a rigid object in terms of the position of its center of mass.

Pre-Test - See Post-Test

Definition (Center of mass)

The center of mass $\overrightarrow{R}$ of a system of particles is defined as the average of their positions, $\vec{r}_i$, weighted by their masses, m_i:

$$\overrightarrow{R} = \frac{\sum m_i r_i}{\sum m_i}$$

For a continuous distribution with mass density $\rho(\vec{r})$ and total mass M, the sum becomes an integral:

$$\begin{aligned}\overrightarrow{R} = \frac{1}{M}\int \vec{r}dm &= \frac{1}{M}\int \rho(\vec{r})\vec{r}dV \\ &= \frac{\displaystyle\int \rho(\vec{r})\vec{r}dV}{\displaystyle\int \rho(\vec{r})dV}\end{aligned}$$

If an object has uniform density then its center of mass is the same as the centroid of its shape.

Examples

- The center of mass of a two-particle system lies on the line connecting the particles (or, more precisely, their individual centers of mass). The center of mass is closer to the more massive object.

  Consider the following Figure 34 below with a and b regarded as particles

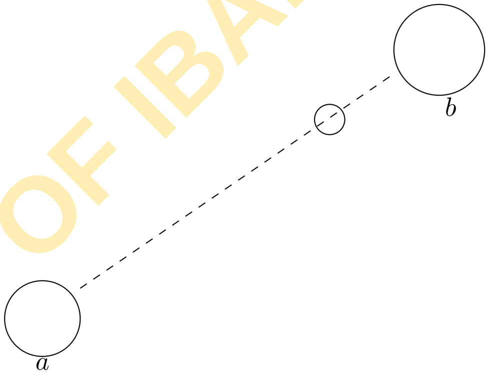


Figure 34:

Now

$$\begin{aligned}\vec{y}_{c.m.} &= 0 \\ \vec{x}_{c.m.} &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2},\end{aligned}$$

$x_1 = 0$, $x_2 = 2.82\ m$
$m_1 = 79.9\ u$, $m_2 = 39.1\ u$

$$x_{c.m.} = \frac{39.1 \times 2.82\ m}{79.9 + 39.1} = 0.93\ m$$

- The center of mass of ring is at the center of the ring (in the air).
- The center of mass of a solid triangle lies on all three medians and therefore at the centroid, which is also the average of the three vertices.

  Consider the following Figure 35 below:

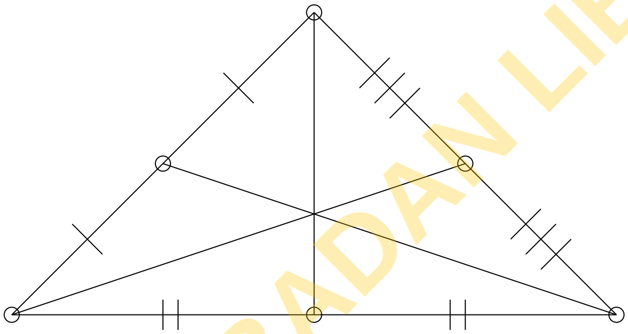


Figure 35:

$$\begin{aligned}\vec{x}_{c.m.} = \frac{\sum m_i x_i}{M} &= \frac{2md + m(d+b) + 4m(d+b)}{7m} \\ &= d + \frac{5b}{7} \\ \vec{y}_{c.m.} = \frac{\sum m_i x_i}{M} &= \frac{2m(0) + m(0) + 4mh}{7m} \\ &= \frac{4h}{7}.\end{aligned}$$

The position vector of the center of mass (c.m.)

$$\vec{r}_{c.m.} = \vec{x}_{c.m.}\hat{x} + \vec{y}_{c.m.}\hat{y} = \left(d + \frac{5}{7}b\right)\hat{x} + \frac{4}{7}h\hat{y}.$$

- The center of mass of a rectangle is at the intersection of the two diagonals.
- In a spherically symmetric body, the center of mass is at the center.

- More generally, for any symmetry of a body, its center of mass will be a fixed point of that symmetry.

Derivation of Center of Mass

Consider first two bodies, with masses m_1 and m_2, and position vector $\vec{r}_1$ and $\vec{r}_2$. Write $M = m_1 + m_2$ for the total mass of the 2-body system, and $\overrightarrow{R}$ for the position vector of the center of mass.

It is reasonable to require, for any system of masses, that the center of mass lie within the convex hull of the system. In particular, for a pair of mass points, this means that the tip of $\overrightarrow{R}$ must lie on the line segment joining the tips of $\vec{r}_1$ and $\vec{r}_2$. By geometry, $\overrightarrow{R} - \vec{r}_1 = k(\vec{r}_2 - \overrightarrow{R})$ for some positive constant k. Taking magnitudes on both sides of this equation, we get $d_1 - kd_2$, where d_1 is the distance from the center of mass to body 1, and d_2 is the distance from the center of mass to body 2. The constant k should obviously depend only on the masses m_1 and m_2, and we will examine the nature of this dependence.

Assuming the total mass M nonzero, it is clear that if $m_2 = 0$, the center of mass should coincide with body 1, and $d_1 = 0$. This means $d_2 = D$, the total distance between the two bodies, and $m_1 = M$. Symmetry demands that these relations remain true when the subscripts 1 and 2 are interchanged everywhere.

The simplest model satisfying these requirements is the linear one, $d_1 = (D/M)m_2$ and $d_2 = (D/M)m_1$. Under this model, we have $k = d_1/d_2 = m_2/m_2$. Therefore, after multiplying our vector equation by m_1, we find that $m_1(\overrightarrow{R} - \vec{r}_1) = m_2(\vec{r}_2 - \overrightarrow{R})$, or $(m_1 + m_2)\overrightarrow{R} = m_1\vec{r}_1 + m_2\vec{r}_2$. Thus,

$$\overrightarrow{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}.$$

Now suppose there is a third body, of mass m_3 and position $\vec{r}_3$. Temporarily break the symmetry between the three bodies, and define the 3-body center of mass as the 2-body center of mass determined by body 3 together with a single body of mass $M_0 = m_1 + m_2$ placed at the center of mass of bodies 1 and 2, whose position vector we now denote by $\overrightarrow{R}_0$. The formula derived above gives

$$\begin{aligned}\overrightarrow{R} = \frac{M_0\overrightarrow{R}_0 + m_3\vec{r}_3}{M_0 + m_3} &= \frac{(m_1 + m_2)\left(\frac{m_1r_1+m_2r_2}{m_1+m_2}\right) + m_3\vec{r}_3}{M_0 + m_3} \\ &= \frac{m_1r_1 + m_2r_2 + m_3r_3}{m_1 + m_2 + m_3}.\end{aligned}$$

Since $\overrightarrow{R}$ turns out to be symmetric in the m_i and r_i, it would not have mattered had we started by combining bodies 2 and 3, or bodies 1 and 3, instead of bodies 1 and 2. This kind of reasoning clearly extends to any number of masses, and yields the formula

$$\overrightarrow{R} = \frac{\sum m_i\vec{r}_i}{\sum m_i}.$$

So our simple model of the 2-body center of mass uniquely and consistently determines the corresponding formula in any number of mass points. Writing $M = m_1 + m_2 + \cdots + m_n$, the above formula for the center of mass may be expressed in the form

$$M\overrightarrow{R} = \sum m_i \vec{r}_i,$$

that is, the sum of the moments of a number of bodies is the momentum of their center of mass. It is this principle that gives precise expression to the intuitive notion that the system as a whole behaves like a mass of M placed at $\overrightarrow{R}$, and justifies our simple linear model of the one-dimensional center of mass.

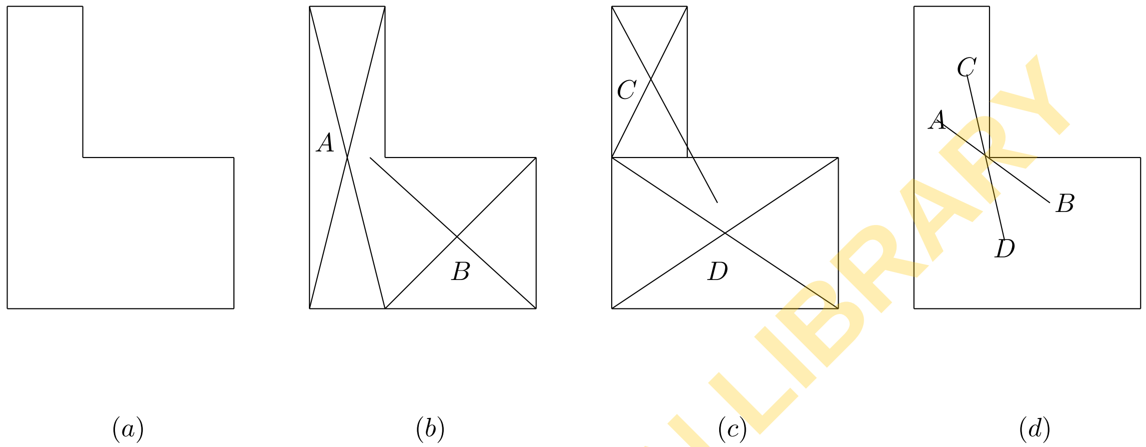
The center of mass is often called the center of gravity because any uniform gravitational field $\vec{g}$ acts on a system as if the mass M of the system were concentrated at the center of mass $\vec{R}$. This is seen in at least two ways:

- The gravitational potential energy of a system is equal to the potential energy of a point particle having the same mass M located at $\overrightarrow{R}$.
- The gravitational torque on a system equals the torque of a force $M\vec{g}$ acting at $\overrightarrow{R}$.
- $\overrightarrow{R} \times M\vec{g} = \sum_i m_i \vec{r}_i \times \vec{g}$.

If the gravitational field acting on a body is not uniform, then the center of mass does not necessarily exhibit these convenient properties concerning gravity.

Locating Center of Mass

This is a method of determining the center of mass of an L-shaped object. (See Figures 36$(a), (b), (c) and (d)$,respectively)



(a) (b) (c) (d)

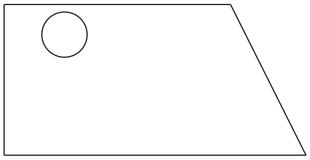
Figure 36:

1. Divide the shape into two rectangles, as shown in Figure 36(b). Find the center of masses of these two rectangles by drawing the diagonals. Draw a line joining the center of masses. The center of mass of the shape must lie on this line AB.
2. Divide the shape into two other rectangles, as shown in Figure 36(c). Find the center of masses of these two rectangles by drawing the diagonals. Draw a line joining the center of masses. The center of mass of the L-shape must lie on this line CD.
3. As the center of mass of the shape must lie along AB and also along CD, it is obvious that it is at the intersection of these two lines, at 0. The point 0 might not lie inside the L-shaped object.

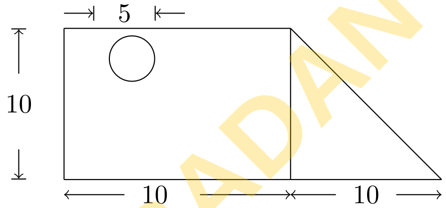
Locating the Center of Mass of a Composite Shape

This method is useful when one wishes to find the location of the centroid or center of mass of an object that is easily divided into elementary shapes, whose centers of mass are easy to find. Here the center of mass will only be found in the x direction. The same procedure may be followed to locate the center of mass in the y direction.

The shape. It is easily divided into a square, triangle, and circle. Note that the circle will have negative area. (See Figure 37(a) below)



(a)



(b)

Figure 37:

We note the coordinates of the individual centroids. (See Figure 37(b) above)

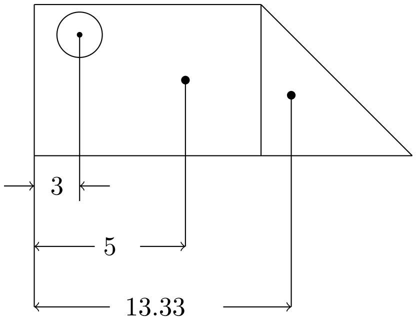


Figure 38:

From equation above the individual centroids.(See Figure 38 above)

$$\frac{3 \times -\pi 2.5^2 + 5 \times 10^2 + 13.33 \times \frac{10^2}{2}}{-\pi 2.5^2 + 10^2 + \frac{10^2}{2}} \simeq 8.5 \text{ units.}$$

The center of mass of this figure is at a distance of 8.5 units form the left corner of the figure.

- Locating the center of mass by tracing around the perimeter of the shape.
  This method can be applied to a shape with an irregular, smooth or complex boundary where other methods are too difficult. It was regularly used by ship builders to ensure the ship would not capsize.

Summary
The position vector of the center-of-mass is the average position vector of the mass of the system

$$\overrightarrow{R} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum m_i \vec{r}_i}{M}$$

In terms of vector components:

$$x = \frac{1}{M}[m_1x_1 + m_2x_2 + \cdots + m_nx_n] = \frac{1}{M}\sum m_ix_i$$

$$y = \frac{1}{M}[m_1y_1 + m_2y_2 + \cdots + m_ny_n] = \frac{1}{M}\sum m_iy_i$$

$$z = \frac{1}{M}[m_1z_1 + m_2z_2 + \cdots + m_nz_n] = \frac{1}{M}\sum m_iz_i$$

$$\overrightarrow{R} = x\hat{x} + y\hat{y} + z\hat{z}.$$

Consider objects with continuous distributions of mass.
Divide body up
into elements of mass $\triangle m_i$ with coordinates x_i, y_i, z_i
and obtain the sum

$$\overrightarrow{R} = \frac{1}{M}\int r\, dm.$$

The total linear momentum of the system is the same as that of a particle of mass M whose motion is that of the center of mass of the system.

Post-Test

1. A square of side $\frac{L}{2}$ is removed from one corner of a square sandwich that has sides of length L. The displacement of the x coordinate of the center of mass (from C to C') is: (see Figure 39)

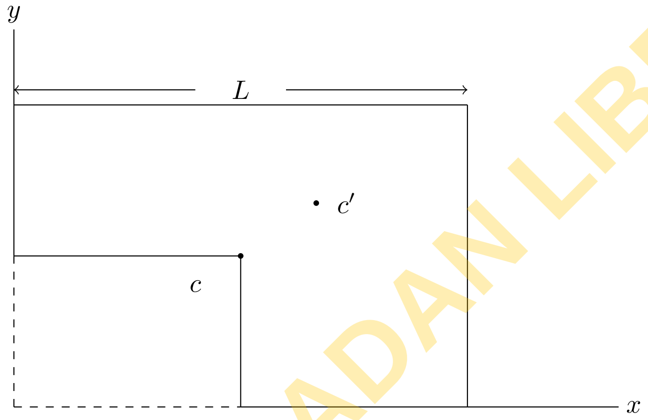


Figure 39:

2. Three particles are placed at the three corners of a square as shown in the figure below.
   What should be the mass of the particle at the remaining corner so that the COM of the system of particles lies at the center of square (see Figure 40 below)

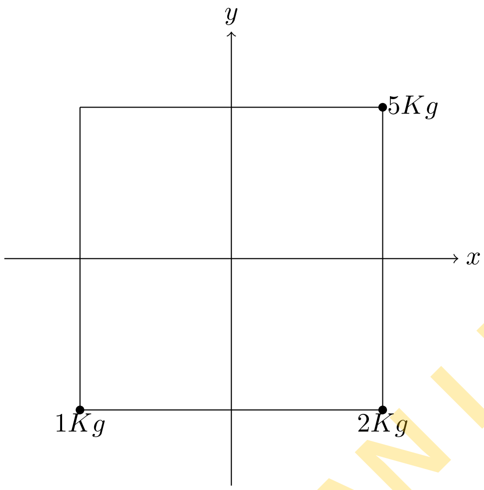


Figure 40:

3. A circular and a square plate are placed in contact as shown in the figure below. If the material and thickness of the two plates are same, then COM of the system of bodies are measured from the point of contact is:(see Figure 41)

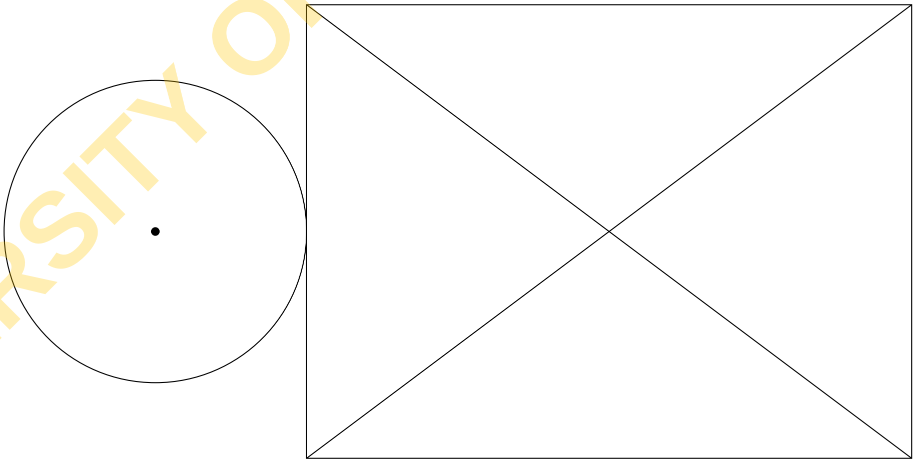


Figure 41:

Supplementary Reading

1. Goldstein, Herbert; Charles Poole, John Safko (2002). Classical Mechanics (3e ed.) Addison Wesley ISBN0-201-65702-3

2. Serway, Raymond; Jewett, John (2004). Physics for Scientists and Engineers (6th ed.). Brooks/Cole. ISBN 0-534-40842-7.

3. Tipler, P. (2004). Physics for Scientists and Engineers: Mechanics, Oscillations and Waves Thermodynamics (5th ed.), W.H. Freeman. ISBN 0-7167-0809-4.

LECTURE 11

Work, Energy and Power

Introduction

In this lecture, we shall analyze the motion of objects from the perspective of work and energy. The effect that work has upon the energy of an object (or system of objects) will be investigated; the resulting velocity and/or height of the object can be predicted from energy information. Thus, we shall focus on the definitions and meanings of such terms as work, mechanical energy, potential energy, kinetic energy, and power.

Objective

At the end of this lecture you should be able to:

- understand the definition of work, including when it is positive, negative, or zero,
- calculate the work done by a specified constant force on an object that undergoes a specified displacement,
- relate the work done by a force to the area under a graph of force as a function of position, and calculate this work in the case where the force is a linear function of position,
- use integration to calculate the work performed by a force $F(x)$ on an object that undergoes a specified displacement in one dimension,

- apply the work-energy theorem,
- understand the concept of a conservative force, potential energy, mechanical energy, total energy, conservation of energy,
- recognize and solve problems that call for application both of conservation of energy and Newton's laws,
- understand the definition of power,
- calculate the power required to maintain the motion of an object with constant acceleration,
- calculate the work performed by a force that supplies constant power, or the average power supplied by a force that performs a specified amount of work.

Pre-Test - See Post-Test

Definition and Mathematics of Work

When a force acts upon an object to cause a displacement of the object, it is said that work was done upon the object. In other words, force and displacement are both vector quantities and they are combined using the dot product to evaluate the mechanical work, a scalar quantity:

$$W = \overrightarrow{F} \cdot \vec{d} = Fd\cos\phi \tag{1}$$

where ϕ is the angle between the force and displacement vector.

In other for this formula to be valid, the force and angle must remain constant. The object's path must always remain on a single, straight line though it may change directions while moving along the line.

Examples

(i) When a force acts rightward upon an object as it is displaced rightward.

$$\begin{array}{c}\vec{d}\\ \longrightarrow\\ \longrightarrow\\ \overrightarrow{F}\end{array} \quad \theta = 0 \ \ \underline{\text{degrees}}$$

In such an instance, the force vector and the displacement vector are in the same direction. Thus the angle between $\overrightarrow{F}$ and $\vec{d}$ is 0 degrees.

(ii) When a force acts leftward upon an object which is displaced rightward.

$$\begin{array}{c}\vec{d}\\ \longrightarrow\\ \longleftarrow\\ \overrightarrow{F}\end{array} \quad \theta = 180 \ \ \underline{\text{degrees}}$$

In such an instance, the force vector and the displacement vector are in the opposite direction. Thus, the angle between $\overrightarrow{F}$ and $\vec{d}$ is 180 degrees.

(iii) When a force acts upward on an object as if is displaced rightward

$$\xrightarrow{\vec{d}} \uparrow \overrightarrow{F} \qquad \theta = 90 \text{ degrees}.$$

In such an instance, the force vector and the displacement vector are at right angles to each other. Thus, the angle between $\overrightarrow{F}$ and $\vec{d}$ is 90 degrees.

The Meaning of Negative Work

The other occurs, when a force acts upon a moving object to hinder a displacement. In such instances, the force acts in the direction opposite the objects motion in other to slow it down. The force doesn't cause the displacement but rather hinders it. These situations involve what is commonly called negative work. Thus,

$$\begin{aligned} \overrightarrow{W} &= \overrightarrow{F} d \cos 180^0 \\ &= -\overrightarrow{F} d. \end{aligned}$$

In situations where the force changes over time, or the path deviated from a straight line, equation (1) is not generally applicable although it is possible to divide the motion into small steps, such that the force and motion are well approximated as being constant for each step, and then to express the overall work as the sum over these steps.
The general definition of mechanical work is given by the following line integral.

$$\overrightarrow{W}_C = \int_C \overrightarrow{F} \cdot d\vec{s} \qquad (2)$$

where
C is the path or curve traversed by the object,
$\overrightarrow{F}$ is the force vector; and
$\vec{s}$ is the position vector.
The expression $\delta\overrightarrow{W} = \overrightarrow{F} \cdot d\vec{s}$ is an inexact differential which means that the calculation of $\overrightarrow{W}_C$ is path-dependent and cannot be differentiated to give $\overrightarrow{F} \cdot d\vec{s}$.

Equation (2) explains how a non-zero force can do zero work. The simplest case is where the force is always perpendicular to the direction of motion, making the integral always zero. This is what happens during circular motion. However, even if the integrand sometimes takes nonzero values, it can still integrate to zero if it is sometimes negative and sometimes positive.

The possibility of a nonzero force doing a zero work illustrated the difference between work and a related quantity, impulse, which is the integral of force over time. Impulse measures change in a body's momentum, a vector quantity sensitive to direction, whereas work considers only the magnitude of the velocity. For instance, as an object in uniform circular motion traverses half of a revolution, its centripetal force does not work, but it transfers a nonzero impulse.

Units of Work

The SI unit of work is the joule (J), which is defined as the work done by a force of one Newton acting over a distance of one meter.

Non-standard Units of Work

$$ft * \text{pound}, \quad Kg * \frac{m}{s^2} * m, \quad Kg * \frac{m^2}{s^2}$$

Practise Exercise

1. Apply the work equation to determine the amount of work done by the applied force in each of the three situations described below.
   - (i) A 100 N force is applied to move a 15 Kg object a horizontal distance of 5 meters at constant speed.
   - (ii) A 100 N force is applied an angle of 30^0 to the horizontal to move a 15 Kg object at a constant speed for a horizontal distance of 5 m.
   - (iii) An upward force is applied to lift a 15 Kg object to a height of 5 meters at constant speed.

Potential Energy

An object can store energy as the result of its position. For example, a drawn bow is able to store energy as the result of its position. When assuming its usual position (i.e., when not drawn), there is no energy stored in the bow. Yet when its position is altered from its usual equilibrium position, the how is able to store energy by virtue of its position. This stored energy of position is referred to as potential energy.
Potential energy is energy which results from position or configuration. An object may have the capacity for doing work as a result of its position in a gravitational field (gravitational potential energy), an electric field (electric potential energy), or a magnetic field (magnetic potential energy). It may have elastic potential energy as a result of a stretched spring or other elastic deformation.

Potential Energy Concept

The potential energy P is equal to the work you must do to move an object from the $P = 0$ reference point to the position r. The reference point at which you assign the value $P = 0$ is arbitrary, so may be chosen for convenience, like choosing the origin of a coordinate system. The force on an object is the negative of the derivative of the potential function P.

$$\begin{aligned} \overrightarrow{F}(x) &= -\frac{dP}{dx} \\ P &= -\int^{r} \overrightarrow{F} \cdot d\vec{r} \end{aligned}$$

This means it is the negative of the slope of the potential energy curve.

Negative Signs in Potential

$\overrightarrow{F}$ in the definition of potential energy is the force exerted by the force field, e.g., gravity, spring force, etc. The potential energy P is equal to the work you must do against that force to move an object from the $P = 0$ reference point to the position r. The force you must exert to move it must be equal but oppositely directed, and that is the source of the negative sign. The force exerted by the force field always tends toward lower energy and will act to reduce the potential energy.

Elastic Potential Energy

The equation is

$$PE_{\text{spring}} = \frac{1}{2} * K * X^2$$

where
K = spring constant.
X = amount of compression (relative to equilibrium position).

Practise Exercise

If a force of 14.7 N is used to drag the loaded cart along the inclined for a distance of 0.90 meters, then how much work is done on the loaded cart?

Kinetic Energy

Kinetic energy is the energy of motion. An object which has motion - whether it be vertical or horizontal motion - has kinetic energy. There are many forms of kinetic energy - vibrational (the energy due to vibrational motion), rotational (the energy due to rotational motion), and translational (the energy due to motion from one location to another). We will focus on translational kinetic energy, which an object has depends upon two variables mass (m) of the object and the speed (v) of the object. The following equation is used to represent the kinetic energy (KE) of an object.

$$KE = \frac{1}{2} * m * v^2$$

where
m = mass of object,
v = speed of object.
The kinetic energy is dependent upon the square of the speed. Kinetic energy is a scalar quantity, having a metric unit of measurement as Joule. Note that 1 Joule is equivalent to 1 $Kg * \frac{m^2}{s^2}$.

Practise Exercise

1. Determine the kinetic energy of a 625 Kg roller coaster car that is moving with a speed of 18.3 m/s.
2. If the roller coaster car is the above problem were moving with twice the speed, then what would be its new kinetic energy?

Mechanical Energy

Mechanical Energy is the energy which is possessed by an object due to its motion or due to its position. Mechanical energy can be either kinetic energy (energy of motion) or potential energy (stored energy of position).

Example

A hammer is a tool which utilize mechanical energy to do work. The mechanical energy of a hammer gives the hammer its ability to apply a force to a nail in order to cause it to be displaced. Because the hammer has mechanical energy (in the form of kinetic energy), it is able to do work on the nail. Mechanical energy is the ability to do work.

Total Mechanical Energy

$$TME = PE + KE$$

As discussed earlier, there are two forms of potential energy - gravitational potential energy and elastic potential energy. Given this fact, the above equation can be rewritten:

$$TME = PE_{\text{grav.}} + PE_{\text{spring}} + KE.$$

Power

Power is the rate at which work is done. It is the work/time ratio. Mathematically, it is computed using the following equation:

$$\text{Power} = \frac{\text{Work}}{\text{time}}$$

The standard metric unit of power is the Watt.

A Watt is equivalent to a Joule/second

$$\begin{aligned} \text{Power} = \frac{\text{Work}}{\text{time}} &= \frac{\text{Force}*\text{Displacement}}{\text{Time}} \\ &= \text{Force} = \frac{\text{Displacement}}{\text{Time}} \\ &= \text{Force}*\text{velocity}. \end{aligned}$$

Remark

This new equation for power reveals that a powerful machine is both strong (big force) and fast (big velocity). A powerful car engine is strong and fast. A machine which is strong enough to apply a big force to cause a displacement in a small mount of time (i.e., a big velocity) is a powerful machine.

Practise Exercise

1. Two MAT 242 students, Taiwo and Kehinde, are in the weightlifting room. Taiwo lifts the 100-pound barbell over his head 10 times in one minute; Kehinde lifts the 100-pound barbell over his head 10 times in 10 seconds. Which student does the most work? Which student delivers the most powerful? Explain your answers.
2. During a MAT 242 lecture, Jack and Jill ran up a hill, Jack is twice as massive as Jill; yet Jill ascends the same distance in half the time. Who did the most work? Who delivered the most power? Explain your answers.

Summary
Work is done when a force acts upon an object to cause a displacement.
Three quantities must be known in order to calculate the amount of work.
Those three quantities are force, displacement and the angle between
the force and the displacement.
Potential energy is the energy which is
stored in an object due to its position relative to some zero position.

An object possesses gravitational potential energy if it is positioned at a height above (or below) the zero height. An object possesses elastic potential energy if it is at a position on an elastic medium other than the equilibrium position.

Kinetic energy is obtained from the equation

$$KE = \frac{1}{2} * m * v^2$$

where

m = mass of object.

v = speed of object.

Total mechanical energy is given by the formula $TME = PE + KE$

$PE_{\text{grav.}} + PE_{\text{spring}} + KE$.

and finally,

$$\text{Power} = \frac{\text{Work}}{\text{time}} = \frac{\text{Force}*\text{Displacement}}{\text{Time}}$$

$$= \text{Force} * \frac{\text{Displacement}}{\text{Time}}$$

$= \text{Force} * \text{velocity}$.

Post-Test

1. A student with a mass of 80.0 Kg runs up three flights of stairs in 12.0 *sec*. The student has gone a vertical distance of 8.0 m. Determine the amount of work done by the student to elevate his body to this height. Assume that her speed is constant.

2. Calculate the work done by a 2.0 N force (directed at a 30^0 angle to the vertical) to move a 500 gram box a horizontal distance of 400 cm across a rough floor at a constant speed of 0.5 m/s.
   (**Hint:** Be cautious with the units).

3. A cart is loaded with a brick and pulled at constant speed along an inclined plane to the height of a seat-top. If the mass of the loaded cart is 3.0 Kg and the height of the seat top is 0.45 meters, then what is the potential energy of the loaded cart at the height of the seat-top?

4. Determine the kinetic energy of a 625 Kg roller coaster car that is moving with a speed of 18.3 m/s.

5. A 900 kg compact car moving at 60 km/hr has approximately 320000 joules of kinetic energy. Estimate its new kinetic energy if it is moving at 30 km/hr.
   (**Hint:** use the kinetic energy equation as a "guide to thinking").

6. Your household's monthly electric bill is often expressed in kilowatt-hours. One kilowatt-hour is the amount of energy delivered by the flow of 1 kilowatt of electricity

for one hour. Use conversion factors to show how many joules of energy you get when you buy 1 kilowatt-hour of electricity.

7. An escalator is used to move 20 passengers every minute from the first floor of a department store to the second. The second floor is located 5.20 meters above the first floor. The average passenger's mass is 54.9 Kg. Determine the power requirement of the escalator in order to move this number of passengers in this amount of time.

Supplementary Reading

1. Serway, Raymond; Jewett, John (2004). Physics for Scientists and Engineers (6th ed.). Brooks/Cole. ISBN 0-534-40842-7.

2. Tipler, Paul (1991). Physics for Scientists and Engineers: Mechanics, (3rd ed., extended version ed.), W.H. Freeman. ISBN 0-87901-432-6.

3. Zitzewitz, Elliot, Haase, Harper, Herzog, Nelson, Nelson, Schuler, Zorn (2005). Physics: Principles and Problems. McGraw-Hill Glencoe, The McGraw-Hill Companies, Inc. ISBN 0-07-845813-7.

LECTURE 12

The Work-Energy Relationship

Introduction

In this lecture, we shall explain and distinguish between internal and external forces. We will explore the quantitative relationship between work and mechanical energy in situations in which there are no external forces doing work as well as in situation involving external forces.

Objective
At the end of this lecture you should be able to:
- identify external and internal forces,
- read descriptions and indicate whether energy is transformed from KE to PE or from PE to KE,
- indicate whether the gain or loss of energy resulted in a change in the object's kinetic energy, potential energy, or both.
- analyze motion situations using the work and energy relationship, and
- solve complex problems by combining the work-energy relationship with the expressions for potential and kinetic energy.

Pre-Test

1. In the following descriptions, indicate whether energy is transformed form KE to PE (or vice-versa) and explain.
   (i) A ball falls from a height of 2 meters in the absence of air resistance.
   (ii) A bungee cord beings to exert an upward force upon a falling bungee jumper.
   (iii) The spring of a dart gun exerts a force on a dart as it is launched from an

initial rest position.

2. In the following descriptions, indicate whether the gain or loss of energy resulted in a change in the object's kinetic energy, potential energy, or both.
   (i) Peter pounds a nail into a block of wood. The hammer head is moving horizontally when it applied force to the nail.
   (ii) The frictional force between highway and tyres pushes backwards on the tyres of a skidding car.
   (iii) A weightlifter applies a force to lift a barbell above his head at constant speed.
3. A 100 kg car travelling with a speed of 25 m/s skids to a stop. The car experiences an 8000 N force of friction. Determine the stopping distance of the car.
4. At the end of the shock wave roller coaster ride, the 600 kg train of cars (including passengers) is slowed from a speed of 20 m/s to a speed of 5 m/s over a distance of 20 meters.
   Determine the braking force required to slow the train of cars by this amount.

Comparing Internal With External Forces

External Forces are forces, which when present and when involved in doing work on objects will change the total mechanical energy of the object. For instance, the applied for, normal force, tension force, friction force, and air resistance force.

When net work is done upon an object by an external force, the total mechanical energy ($KE + PE$) of that object is changed. If the work is positive work, then the object will gain energy. If the work is negative work, then the object will lose energy. The gain or loss in energy can be in the form of potential energy, kinetic energy, or both. Under such circumstances, the work which is done will be equal to the change in mechanical energy of the object. Because external forces are capable of changing the total mechanical energy of an object, they are sometimes referred to as non-conservative forces.

Internal Forces are forces, which can never change the total mechanical energy of an object, but rather can only transform the energy of an object from potential energy to kinetic energy (or vice versa). For instance, the gravity forces, magnetic force, electrical force, and spring force. When the only forces doing work are internal forces, energy changes forms - from kinetic to potential (or vice versa); yet the total amount of mechanical is conserved. Because internal forces are capable of changing the form of energy without changing the total amount of mechanical energy, they are sometimes referred to as conservative forces.

Note:

A horizontal force can never change the potential energy of an object. Horizontal forces cannot cause vertical displacements. The only means by which an external or non-conservative force can contribute to a potential energy change is if the force has a vertical component. Potential energy changes are the result of height changes and only a force with a vertical component can cause a height change.

Analysis of Situations Involving External Forces

The quantitative relationship between work and mechanical energy is expressed by the following equation:

$$TME_I + W_{EXT} = TME_F$$

The equation states that the initial amount of total mechanical energy (TME_I) plus the work done by external forces (W_{EXT}) is equal to the final amount of total mechanical energy (TME_F).

Notes:

First, the mechanical energy can be either potential energy (in which case it could be due to springs or gravity) or kinetic energy. Given this fact, the above equation can be rewritten as

$$KE_I + PE_I + W_{EXT} = KE_F + PE_F$$

The second note which should be made about the above equation is that the work done by external forces can be a positive or a negative work term.

Examples (nonconservative forces)

1. Consider a weightlifter who applies an upwards force (say 1000 N) to a barbell to displace it upwards a given distance (say 0.25 meters) at a constant speed. The initial energy plus the work done by the external force equals the final energy. If the barbell begins with 1500 joules of energy (this is just a made up value) and the weightlifter does 250 joules of work ($F * d *$ cosine of angle $= 1000\ N * 0.25\ m *$cosine 0 degrees $= 250\ J$), then the barbell will finish with 1750 joules of mechanical energy. The final amount of mechanical energy (1750 J) is equal to the initial amount of mechanical energy (1500 J) plus the work done by external forces (250 J). That is, $TME_F = 1500\ J + 250\ J = 1750\ J$.

2. Now consider a baseball catcher who applies a rightward force (say 6000 N) to a leftward moving baseball to bring it from a high speed to a rest position over a given distance (say 0.10 meters). The initial energy plus the work done by the external force equals the final energy. If the ball begins with 605 joules of energy (this is just another made up value), and the catcher does Work $= 6000\ N * 0.10\ m * \cos 180^0 = -600\ J$.

$$\begin{aligned} TME_F &= TME_I + W_{EXT} \\ &= 605\ J + (-600\ J) = 5\ J \end{aligned}$$

3. Now consider a car which is skidding from a high speed to a lower speed. The force of friction between the tyres and the road exerts a leftward force (say 8000 N) on

the rightward moving car over a given distance (say 30 m). If the car begins with 320,000 joules of energy (this is just another made up value), and the friction force does

$$8000\ N * 30\ m * \cos 180^0 = -240,000\ J.$$

$$\begin{aligned} TME_F &= 320,000\ J + (-240,000) J \\ &= -80000\ J. \end{aligned}$$

4. Finally, consider a cart being pulled up an inclined plane at constant speed by a student during a lecture break. The applied force n the cart (say 18 N) is directed parallel to the incline to cause the cart to be displaced parallel to the incline for a given displacement (say 0.7 m). If the cart begins with 0 Joules of energy (that is just another made up value), and the student does $18\ N * 0.70\ m * \cos 0^0 = 12.6\ J$.

$$\begin{aligned} TME_F &= TME_I + W_{EXT} \\ &= 0J + 12.6\ J = 12.6\ J \end{aligned}$$

Analysis of Situations in which Mechanical Energy is Conserved

The quantitative relationship between work and mechanical energy in situations in which there are no external forces doing work in expressed by the following equation:

$$KE_I + PE_I = KE_F + PE_F$$

Remark

In these situations, the sum of the kinetic and potential energy is everywhere the same. As the potential energy is increased due to the stretch/compression of a spring or an increase in its height above the earth, the kinetic energy is decreased due to the object slowing down. As the potential energy is decreased due to the return of a spring to its rest position or a decrease in height above the earth, the kinetic energy is increased due to the object speeding up. We would say that energy is transformed or changes its form from kinetic energy to potential energy (or vice versa), yet the total amount present is conserved - that is, always the same.

Practice Exercise

1. Consider the falling and rolling motion of the ball in the following two resistance-free situations. In one situation, the ball falls off the top of the platform to the floor. In the other situation, the ball rolls from the top of the platform along the staircase - like pathway to the floor. For each situation, indicate what type of forces are doing work upon the ball. Indicate whether the energy of the ball is conserved and explain why.

2. Some driver's license exams have the following question:
   A car moving 50 km/hr skids 15 meters with locked brakes. How far will the car skid with locked brakes if it is moving at 150 km/hr.

3. An object which weighs 10 N is dropped from rest from a height of 4 meters above the ground. When it has free-fallen 1 meter its total mechanical energy with respect to the ground is - - - - - - - - - ?

4. During a certain time interval, a $20 - N$ object free-falls 10 meters. The object gains ——— Joules of kinetic energy during this interval.

Summary

To understand the work-energy relation, you must understand the two types of forces, namely internal forces and external forces.
There is a relationship between work and mechanical energy change.
Whenever work is done upon an object by an external or nonconservative force, there will be a change in the total mechanical energy of the object.
If only internal forces are doing work (no work done by external forces), there is no change in total mechanical energy; the total mechanical energy is said to be "conserved". The quantitative relationship between work and the two forms of mechanical energy is expressed by the following equation:
$KE_I + PE_I + W_{EXT} = KE_F + PE_F$

Post-Test(See Pre-Test)

Supplementary Reading

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LECTURE 13

Gradient, Divergence and Curl.

Introduction

This lecture develops the functions of vector arguments which arise in practical applications and are known as the gradient, the divergence and the curl.

Objective
At the end of this lecture you should be able to:
- recite basic identities regarding the "del" operator ∇ applied to various combinations of scalar and vector fields.
- Solve problems involving gradients, divergence, and curl, using the cylindrical or spherical or cartesian or orthogonal systems.

Pre-Test - See Post-Test

Field in Cartesian Coordinates

Partial derivatives

$$\partial_i = \frac{\partial}{\partial x_i}$$

Nabla (Del) operator

$$\nabla = \vec{i}_k \partial_k = \vec{i}_1 \partial_1 + \vec{i}_2 \partial_2 + \vec{i}_3 \partial_3$$

Gradient

$$\vec{\text{grad}}\ f = \nabla f = \vec{i}_k \partial_k f$$

$$(\vec{\text{grad}}\ f)_i = \partial_i f$$

Example

If $f(x,y,z) = 3x^2y - y^3z^2$, find $\triangle f$ (or $\vec{\text{grad}} f$) at the point $(1,-2,-1)$

$$\begin{aligned}\nabla f &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(3x^2y - y^3z^2) \\ &= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2) \\ &= 6xy\vec{i} + (3x^2 - 3y^2z^2)\vec{j} - 2y^3z\vec{k} \\ &= 6(1)(-2)\vec{i} + 3(1)^2 - 3(-2)^2(-1)^2\vec{j} - 2(-2)^3(-1)\vec{k} \\ &= -12\vec{i} - 9\vec{j} - 16\vec{k}.\end{aligned}$$

Practice Exercise

If $f = x^2z + e^{y/x}$ and $g = 2z^2y - xy^2$, find (a) $\nabla(f+g)$ and $\nabla(fg)$ at the point $(1,0,-2)$.

Directional Derivative

$$\frac{df}{ds} = \frac{d\vec{r}}{ds} \cdot \vec{\text{grad}} f) = \vec{u} \cdot \vec{\text{grad}} f)$$

Flow lines

$$\begin{aligned}\frac{d\vec{r}}{dt} &=^{\beta} \vec{F} \\ \frac{dx_1}{F_1} &= \frac{dx_2}{F_2} = \frac{dx_3}{F_3}\end{aligned}$$

Example

If $\vec{u} = x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}$, find $\nabla \cdot \vec{u}$ at the point $(1,-1,1)$.

$$\begin{aligned}\nabla \cdot \vec{u} &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}) \\ &= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z) \\ &= 2xz - 6y^2z^2 + xy^2 \\ &= 2(1)(1) - 6(-1)^2(1)^2 + (1)(-2)^2 = -3 \ \text{ at } (1,-1,1).\end{aligned}$$

Practice Exercise

If $\vec{u} = 2x^2\vec{i} - 3yz\vec{j} + xz^2\vec{k}$ and $f = 2z - x^3y$, find $\vec{u} \cdot \nabla f$ at the point $(1,-1,1)$.

Divergence

$$\text{div}\overrightarrow{F} = \nabla \cdot \overrightarrow{F} = \partial_i F_i$$

Practice Exercise

Evaluate div$(2x^2z\vec{i} - xy^2z\vec{j} + 3yz^2\vec{k})$

Curl

$$\begin{aligned}
\mathrm{C\vec{u}rl}\overrightarrow{F} &= \begin{vmatrix} i_1 & i_2 & i_3 \\ \\ \partial_1 & \partial_2 & \partial_3 \\ \\ f_1 & F_2 & F_3 \end{vmatrix} \\
(\mathrm{C\vec{u}rl}\overrightarrow{F})_i &= \epsilon_{ijk}\partial_j F_k \\
\mathrm{C\vec{u}rl}\overrightarrow{F} &= \nabla \times \overrightarrow{F} = \vec{i}_{ijk}\partial_j F_k
\end{aligned}$$

Example:

If $\vec{u} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$, find $\nabla \times \vec{u}$ (or $\overrightarrow{\mathrm{Curl}\vec{u}}$) at the point $(1, -1, 1)$.

$$\begin{aligned}
\nabla \times \vec{u} &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times (xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}) \\
&= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} \\
&= \left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial y}(-2x^2yz)\right]\vec{i} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\vec{j} \\
&\quad + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]\vec{k} \\
&= (2x^4 + 2x^2y)\vec{i} + 3xz^2\vec{j} - 4xyz\vec{k} \\
&= 3\vec{j} + 4\vec{k} \ \text{ at } \ (1, -1, 1).
\end{aligned}$$

Practice Exercise

If $\vec{u} = 2xz^2\vec{i} - yz\vec{j} + 3xz^3\vec{k}$, find $\nabla \times \vec{u}$ at the point $(1, 1, 1)$.

Laplacian

$$\begin{aligned}
\triangle &= \mathrm{div}\ \overrightarrow{\mathrm{grad}} = \nabla^2 = \nabla \cdot \nabla \\
\triangle &= \partial_i\partial_i = \partial_1^2 + \partial_2^2 + \partial_3^2
\end{aligned}$$

Vector identities:

$$\begin{aligned}
\nabla \times (\nabla \times \overrightarrow{F}) &= -\triangle\overrightarrow{F} + \nabla(\nabla \cdot \overrightarrow{F}) \\
\overrightarrow{\mathrm{Curl}}\ \overrightarrow{\mathrm{Curl}}\overrightarrow{F} &= -\triangle\overrightarrow{F} + \overrightarrow{\mathrm{grad}}\ \mathrm{div}\overrightarrow{F}
\end{aligned}$$

$\nabla \times \nabla\varphi = 0$,

$\overrightarrow{\mathrm{Curl}}\ \overrightarrow{\mathrm{grad}} = 0$

$\nabla\cdot(\nabla\times\overrightarrow{F})=0$
div $\overrightarrow{\mathrm{Curl}}=0$

$$\begin{aligned}
\nabla(fg) &= (\nabla f)g+f(\nabla g)\\
\nabla(f\overrightarrow{F}) &= (\nabla f)\cdot\overrightarrow{F}+f\nabla\overrightarrow{F}\\
\nabla\times(f\overrightarrow{F}) &= (\nabla f)\times\overrightarrow{F}+f(\nabla\times\overrightarrow{F})\\
\nabla(\vec{u}\times\vec{v}) &= \vec{v}\cdot(\nabla\times\vec{u})-\vec{u}\cdot(\nabla\times\vec{v})\\
\nabla f(\varphi) &= \frac{df}{d\varphi}\nabla\varphi\\
\nabla\cdot\overrightarrow{F}(\varphi) &= \nabla\varphi\cdot\frac{d\overrightarrow{F}}{d\varphi}\\
\nabla\times\overrightarrow{F}(\varphi) &= \nabla\varphi\times\frac{d\overrightarrow{F}}{d\varphi}\\
\nabla\vec{r} &= 3\\
\nabla\times\vec{r} &= 0\\
r &= |\vec{r}|=\sqrt{x_1^2+x_2^2+x_3^2}\\
\nabla r &= \frac{\vec{r}}{r},\quad |\nabla r|=1\\
\nabla f(r) &= \frac{df}{dr}\,\frac{\vec{r}}{r}\\
\nabla\cdot\overrightarrow{F}(r) &= \frac{\vec{r}}{r}\cdot\frac{d\overrightarrow{F}}{dr}\\
\nabla\times\overrightarrow{F}(r) &= \frac{\vec{r}}{r}\times\frac{d\overrightarrow{F}}{dr}\\
(\overrightarrow{F}\cdot\nabla)\vec{r} &= \overrightarrow{F}.
\end{aligned}$$

Fields in Orthogonal Coordinate System (in orthonormal basis e_i)

Vector components

$$\begin{aligned}
F_i &= \vec{F}\cdot e_i\\
\overrightarrow{\mathrm{grad}f} &= e_1\frac{1}{h_1}\frac{\partial}{\partial q^1}f+e_2\frac{1}{h_2}\frac{\partial}{\partial q^2}f+e_3\frac{1}{h_3}\frac{\partial}{\partial q^3}f\\
\mathrm{div}\vec{F} &= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q^1}(h_2h_3F_1)+\frac{\partial}{\partial q^2}(h_3h_1F_2)+\frac{\partial}{\partial q^3}(h_1h_2F_3)\right\}\\
\overrightarrow{\mathrm{Curl}}\vec{F} &= e_1\frac{1}{h_2h_3}\left[\frac{\partial}{\partial q^2}(h_3F_3)-\left[\frac{\partial}{\partial q^3}(h_2F_2)\right]\right]+e_2\frac{1}{h_3h_1}\left[\frac{\partial}{\partial q^3}(h_1F_1)-\left[\frac{\partial}{\partial q^1}(h_3F_3)\right]\right]\\
&\quad+e_3\frac{1}{h_2h_2}\left[\frac{\partial}{\partial q^1}(h_2F_2)-\left[\frac{\partial}{\partial q^2}(h_1F_1)\right]\right]
\end{aligned}$$

$$\triangle f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}f\right)+\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}f\right)+\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}f\right)\right\}$$

Cylindrical Coordinates:

$$\begin{aligned}
\overrightarrow{\text{grad}} f &= e_\rho \partial_\rho f + e_\varphi \frac{1}{\rho}\partial_\varphi f + \rho_z \partial_z f \\
\text{div}\overrightarrow{F} &= \frac{1}{\rho}\partial_\rho(\rho F_\rho) + \frac{1}{\rho}\partial_\varphi F_\varphi + \partial_z F_z \\
\overrightarrow{\text{Curl}}\overrightarrow{F} &= \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\varphi & e_z \\ \partial_\rho & \partial_\varphi & \partial_z \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix} \\
\triangle f &= \frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \frac{1}{\rho^2}\partial_\varphi^2 f + \partial_z^2 f
\end{aligned}$$

Spherical Coordinates

$$\begin{aligned}
\overrightarrow{\text{grad}} f &= e_r \partial_r f + e_\theta \frac{1}{r}\partial_\theta f + e_\varphi \frac{1}{r\sin\theta}\partial_\varphi f \\
\text{div}\overrightarrow{F} &= \frac{1}{r^2}\partial_r(r^2 F_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\partial_\varphi F_\varphi \\
\overrightarrow{\text{Curl}}\vec{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} e_r & re_\theta & r\sin\theta e_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ F_r & rF_\theta & r\sin\theta F_\varphi \end{vmatrix} \\
\triangle f &= \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 f.
\end{aligned}$$

Summary
We give formula of a gradient, divergence, curl, Laplacian in Cartesian, orthogonal, cylindrical and spherical coordinates system.

Post-Test

1. Find the values of the constants a, b, c so that the directional derivative of $f = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a directional parallel to the z-axis.

2. If $\vec{u} = 3xyz^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$ and $f = 3x^2 - yz$, find (a) $\nabla \cdot \vec{u}$, (b) $\vec{u} \cdot \nabla f$, (c) $\nabla \cdot (f\vec{u})$, (d) $\nabla \cdot (\nabla f)$, at the point $(1, -1, 1)$.

3. Prove that $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$.

4. If $f = 3x^2y$, $g = xz^2 - 2y$, evaluate $\text{grad}[(\text{grad}f) \cdot (\text{grad}g)]$.

5. If $\vec{u} = \frac{\vec{r}}{r}$, find grad div$\vec{u}$.

6. Show that $\vec{u} = (2x^2 + 8xy^2z)\vec{i} + (3x^3y - 3xy)\vec{j} - (4y^2z^2 + 2x^3z)\vec{k}$ is not solenoidal but $\vec{v} = xyz^2\vec{u}$ is solenoidal.

7. If $\vec{u} = yz^2\vec{i} - 3xz^2\vec{j} + 2xy = \vec{k}$, $\vec{v} = 3x\vec{i} + 4z\vec{j} - xy\vec{k}$ and $f = xyz$, find
   - (a) $\vec{u} \times (\nabla f)$,
   - (b) $(\vec{u} \times \nabla)f$,
   - (c) $(\nabla \times \vec{u}) \times \vec{v}$,
   - (d) $\vec{v} \cdot \nabla \times \vec{u}$.

8. Prove $\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})$.

9. If $\vec{u}(x, y, z)$ is an invariant differentiable vector field with respect to a rotational of axes, prove that (a) div$\vec{u}$ and (b) Curl$\vec{u}$ are invariant scalar and vector fields respectively under the transformation.

10. Show that the Laplacian operator is invariant under a rotation.

Supplementary Reading

1. Hand, Louis N.; Finch, Janet D. Analytical Mechanics. Cambridge University Press.

2. Lanczos, Cornelius (1970). The variational Principles of Mechanics. Toronto: University of Toronto Press. ISBN 0-8020-1743-6.

3. Murray R.S. (1974). Schaum's Outline of Theory and Problems of Vector Analysis and an introduction to Tensor Analysis. McGraw-Hill, Inc.

LECTURE 14

The Divergence Theorem, Stokes' Theorem and Related Integral Theorems

Introduction

In this lecture, we shall discuss vector fields with the study of Green, Gauss and Stokes Theorem and applications of vector field theory.

Objective
At the end of this lecture you should be able to:
- evaluate the line integral of the vector field over a path, given a vector field and the three-dimensional path,
- state the Divergence Theorem,
- apply these theorems to the computation of line integrals, surface integrals and volume integrals, and
- solve problems involving line integrals, circulation of vector field, the unit normal, the surface element, surface integral of a scalar field, flux of a vector field, flux of a curl, flux of a tensor field, Divergence of a tensor and curl of an antisymmetric 2-tensor.

Pre-Test - See Post-Test

Integrals

Parametrization of a curve C:

$$\vec{r} = \vec{r}(t), \quad a \leq t \leq b$$

Line Integrals

$$\int_C \overrightarrow{F} \cdot d\vec{r} = \int_a^b \overrightarrow{F}(x(t)) \cdot \frac{d\vec{r}}{dt} dt$$

Circulation of vector field along a closed contour

$$\oint_C \overrightarrow{F} \cdot d\vec{r}$$

Parametrization of a surface S

$$\vec{r} = \vec{r}(u, v), \quad a \leq u \leq b, \quad c \leq v \leq d$$

Unit normal

$$\vec{n} = \frac{\partial_u \vec{r} \times \partial_v \vec{r}}{|\partial_u \vec{r} \times \partial_v \vec{r}|}$$

For a surface given by

$$f(\vec{r}) = C$$

the Unit Normal is

$$\vec{n} = \frac{\nabla f}{|\nabla f|}$$

Surface Element

$$\begin{aligned} d\vec{s} &= \vec{n} ds = \partial_u \vec{r} \times \partial_v \vec{r} du dv \\ ds &= |\partial_u \vec{r} \times \partial_v \vec{v}| du dv \end{aligned}$$

For a surface given by

$$z = f(x, y)$$

$$a \leq x \leq b, \quad y_1(x) \leq y \leq y_2(x)$$

the Surface Element is

$$ds = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2} dy dx$$

Surface Integral of a scalar field

$$\begin{aligned} \int\int_S \varphi ds &= \int_c^d \int_a^b \varphi(\vec{r}(u, v)) |\partial_u \vec{r} \times \partial_v \vec{r}| du dv \\ &= \int_a^b \int_{y_1(x)}^{y_2(x)} \varphi(x, y, z(x, y)) \cdot \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2} dy dx \end{aligned}$$

Flux of a Vector Field $\overrightarrow{F}$ through the surface S

$$\int\int_S \overrightarrow{F} \cdot d\vec{S} = \int\int_S \overrightarrow{F} \cdot \vec{n} dS$$

Line Integral of a Gradient

$$\int_P^Q \overrightarrow{\text{grad}}\varphi \cdot d\vec{r} = \varphi(Q) - \varphi(P)$$

Circulation of a Gradient along a closed contour

$$\oint_C \overrightarrow{\text{grad}}\varphi \cdot d\vec{r} = 0$$

Gauss (Divergence) Theorem

$$\int\int_D\int \text{div}\overrightarrow{F}\,dV = \int\int_{\partial D} \vec{F} \cdot d\vec{S}$$

∂D is a closed surface, which is the boundary of the solid region D.

Green's Theorem

$$\int\int_S (\partial_1 F_2 - \partial_2 F_1)dx_1 dx_2 = \oint_{\partial S} (F_1 dx_1 + F_2 dx_2)$$

∂S is a closed plane curve, which is the boundary of the region S in the x_1x_2-plane.

Stokes' Theorem

$$\int\int_S \overrightarrow{\text{Curl}}\overrightarrow{F} \cdot d\overrightarrow{S} = \oint_{\partial S} \overrightarrow{F} \cdot d\vec{r}$$

∂S is a closed space curve, which is the boundary of the surface S.

$$\int\int_S \overrightarrow{\text{Curl}}\overrightarrow{F} \cdot d\overrightarrow{S} = 0$$

Tensor Fields

Flux of a Tensor Field

$$\int\int_S T_{ik}n_k dS, \int\int_S T_{ik}n_i dS$$

Divergence of a Tensor (in Cartesian Coordinates)

$$\partial_i T_{ik}, \quad \partial_k T_{ik}$$

Directional Derivative (in Cartesian Coordinates)

$$\frac{dT_{ik}}{dS} = \frac{dx_j}{dS}\partial_j T_{ik}$$

Analog of Curl of an antisymmetric 2-tensor

$$\begin{aligned} 3\epsilon_{ijk}\partial_i A_{jk} &= \partial_1 A_{23} + \partial_2 A_{31} + \partial_3 A_{13} \\ &= \partial_1 \bar{A}_1 + \partial_2 \bar{A}_2 + \partial_3 \bar{A}_3 \\ &= \text{div}\bar{A}. \end{aligned}$$

Examples

1. Verify Green's theorem in the plane for

$$\oint_C (xy+y^2)dx + x^2dy$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.
Solution
$y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$. Along $y = x^2$, the line integral equals

$$\int_0^1 ((x)(x^2)+x^4)dx + (x^2)(2x)dx = \int_0^1 (3x^3+x^4)dx = \frac{19}{20}$$

Analog $y = x$ from $(1,1)$ to $(0,0)$ the line integral equals

$$\int_1^0 ((x)(x)+x^2)dx + x^2dx = \int_1^0 3x^2dx = -1$$

Then the required line integral $= \dfrac{19}{20} - 1 = \dfrac{-1}{20}$

$$\begin{aligned} \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy &= \int\int_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dxdy \\ &= \int\int_R (x-2y)dxdy = \int_{x=0}^1 \int_{y=x^2}^x (x-2y)dydx \\ &= \int_0^1 \left[\int_{x^2}^x (x-2y)dy\right] dx = \int_0^1 (xy-y^2)\Big|_{x^2}^x dx \\ &= \int_0^1 (x^4-x^3)dx = -\frac{1}{20} \end{aligned}$$

so that the theorem is verified.

2. Show that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\oint_C xdy - ydx.$$

Solution
In Green's theorem, put $M = -y$, $N = x$. Then

$$\begin{aligned} \oint_C xdy - ydx &= \int\int_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dxdy \\ &= 2\int\int_R dxdy \\ &= 2A \end{aligned}$$

where A is the required area. Thus

$$A = \frac{1}{2}\oint xdy - ydx.$$

3. Evaluate
$$\int\int_S \overrightarrow{F} \cdot \vec{n} dX,$$
where $\overrightarrow{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

   Solution

   By the divergence theorem, the required integral is equal to

$$\begin{aligned} & \int\int\int_V \nabla \cdot \overrightarrow{F} dV = \int\int\int_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV \\ = & \int\int\int_V (4z - y) dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z - y) dzdydx \\ = & \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz\Big|_{z=0}^{1} dydx = \int_{x=0}^{1}\int_{y=0}^{1} (2 - y) dydx = \frac{3}{2}. \end{aligned}$$

4. Evaluate
$$\int\int_S \vec{r} \cdot \vec{n} dS,$$
where S is a closed surface.
Solution

   By the divergence theorem,

$$\begin{aligned} \int\int_S \vec{r} \cdot \vec{n} dS &= \int\int\int_V \nabla \cdot \vec{r} dV \\ &= \int\int\int_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) dV \\ &= \int\int\int \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dV \\ &= 3\int\int\int_V dV = 3V \end{aligned}$$

   where V is the volume enclosed by S.

5. If $\overrightarrow{F} = \vec{x}$ and V is a sphere of radius R centred at the origin, verify the divergence theorem.
Solution

$$\begin{aligned}\int\int\int_V \nabla\cdot\overrightarrow{F}dV &= \int\int\int_V \nabla\cdot\vec{x}dV = 3\int\int\int_V dV\\ &= 3\times\frac{4}{3}\pi R^3 = 4\pi R^3\end{aligned}$$

Alternatively,

$$\int\int_S \overrightarrow{F}\dot{d}\overrightarrow{S} = \int\int_S \vec{x}\cdot\vec{n}dS,$$

but $\vec{n} = \vec{x}/R$ so

$$\begin{aligned}\int\int_S \overrightarrow{F}\cdot d\overrightarrow{S} &= \int\int_S \frac{\vec{x}\cdot\vec{x}}{R}dS\\ &= \int\int_S \frac{R^2}{R}dS = R\int\int_S dS\\ &= R\times 4\pi R^2 = 4\pi R^3,\end{aligned}$$

as expected.

6. **Example on Divergence Theorem's Application Gauss's Law**

If we integrate the first of Maxwell's laws $\nabla\cdot\overrightarrow{E} = \frac{1}{\epsilon_0}\rho$ over a volume V, we have

$$\int\int\int_V \nabla\cdot\overrightarrow{E}dV = \frac{1}{\epsilon_0}\int\int\int_V \rho dV.$$

Using the divergence theorem on the left-hand side gives,

$$\int\int_S \overrightarrow{E}\cdot d\overrightarrow{S} = \frac{1}{\epsilon_0}\int\int\int_V \rho dV, \qquad (1)$$

Flux of $\overrightarrow{E}$ over $S = \frac{1}{\epsilon_0}\times$ Total charge enclosed by S.
We can use the relationship (1) to deduce the electric field generated by a point charge e located at the origin.

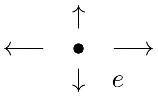

Figure: 42

A charge e is located at the origin of a spherical polar coordinate system (see Figure 42 above). On any sphere of constant radius centred at the origin, we expect the electric field to be constant and parallel to the outer unit normal of the sphere.

By symmetry the electric field must act only in the radial direction and should depend only on the distance from the point charge, $r = |\vec{x}|$, so

$$\overrightarrow{E} = (E_r(r), 0, 0)$$

in spherical polars.

Let V be a sphere of radius R centred at the point charge, then r is constant on the surface S and hence $\overrightarrow{E}$ is also constant on the surface. Moreover, the direction of the electric field is parallel to the outer unit normal to the surface and so

$$\begin{aligned} \int\!\!\int_S \overrightarrow{E} \cdot d\overrightarrow{S} &= \int\!\!\int_S E_r(R)dS = E_r(R)\int\!\!\int_S dS \\ &= E_r(R)4\pi R^2. \end{aligned}$$

The total charge enclosed by the sphere is just e, however, and from equation (1)

$$E_r(R)4\pi R^2 = \frac{e}{\epsilon_0} \Rightarrow E_r(R) = \frac{e}{4\pi R^2 \epsilon_0}$$

the inverse-square law for the electric field is then given by

$$\overrightarrow{E} = \frac{e}{4\pi\epsilon_0}\frac{\vec{x}}{r^3}.$$

7. **Example**

Verify Stokes' theorem for $\vec{u} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution

The boundary C of S is a circle in the xy-plane of radius one and centre at the origin. Let $x = \cos t$, $y \sin t$,. $z = 0|$, $0 \le t < 2\pi$ be parametric equations of C. Then

$$\begin{aligned} \oint_C \vec{u} \cdot d\vec{r} &= \oint_C (2x - y)dx - yz^2dy - y^2zdz \\ &= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt = \pi \end{aligned}$$

Also

$$\nabla \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = \vec{k}$$

Then

$$\int_S (\nabla \times \vec{u}) \cdot \vec{n} dS = \int\int_S \vec{k} \cdot \vec{n} dS = \int\int_R dxdy$$

Since $\vec{n} \cdot \vec{k} dS = dxdy$ and R is the projection of S on the xy-plane. The last integral equals

$$\begin{aligned} \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx &= 4 \int_0^1 \int_0^{\sqrt{1-x^2}} dydx \\ &= 4 \int_0^1 \sqrt{1-x^2} dx = \pi \end{aligned}$$

and Stokes' theorem is verified.

8. **Example**

If $\overrightarrow{F} = (x^2, z^2, -y^2)$, evaluate the integral

$$I = \int\int_S \nabla \times \overrightarrow{F} \cdot d\overrightarrow{S}$$

where $\overrightarrow{S}$ is the surface of a hemisphere of radius 2, centred at $(0,0,0)$ and for which $z = 0$.

Solution

By Stokes' theorem,

$$I = \int\int_S \nabla \times \overrightarrow{F} \cdot d\overrightarrow{S} = \oint_C \overrightarrow{F} \cdot d\vec{x},$$

where C is the circle $x^2 + y^2 = 4$, in the plane $z = 0$. On the circle C,

$$\begin{aligned} \overrightarrow{F} \cdot d\vec{x} &= (x^2, z^2 = 0, -y^2) \cdot (dx, dy, dz = 0) \\ &= x^2 dx, \end{aligned}$$

so

$$\oint_C \overrightarrow{F} \cdot d\vec{x} = \oint_C x^2 dx$$

On the circle $x = 2\cos\theta$, where $0 \leq \theta \leq 2\pi$ and so $dx = -2\sin\theta d\theta$ and then

$$\begin{aligned} I &= \int_0^{2\pi} 4\cos^2\theta \times (-2\sin\theta) d\theta \\ &= 8 \left[\frac{\cos^3\theta}{3} \right]_0^{2\pi} = 0. \end{aligned}$$

9. **Example on Stokes' theorem's application: Faraday's Law**

Integrating the third of Maxwell's laws $\nabla \times \overrightarrow{E} = -\dfrac{\partial \overrightarrow{B}}{\partial t}$ over a surface S we obtain

$$\int\int_S \nabla \times \overrightarrow{E} \cdot d\overrightarrow{S} = -\int\int_S \frac{\partial \overrightarrow{B}}{\partial t} \cdot d\overrightarrow{S}.$$

Using Stokes' theorem on the left-hand side and assuming that the surface remains fixed yields

$$\oint_C \overrightarrow{E} \cdot d\vec{x} = -\frac{d}{dt}\int\int_S \overrightarrow{B} d\overrightarrow{S},$$

$$\text{E.m.f. round circuit} \quad = \quad -\text{rate of change of flux of magnetic field through a surface spanning the circuit.}$$

10. **Example on Stokes' theorem's application: Ampère's law**

If we have a steady current flowing around a circuit C, we can integrate the Maxwell's law $\nabla \times \overrightarrow{B} = \mu_0 \overrightarrow{J}$ over a surface S with boundary C to obtain

$$\int\int_S \nabla \times \overrightarrow{B} \cdot d\overrightarrow{S} = \mu_0 \int\int_S \overrightarrow{J} \cdot d\overrightarrow{S}.$$

Using Stokes' theorem on the left-hand side gives

$$\oint_C \overrightarrow{B} \cdot d\vec{x} = \mu_0 \int\int_S \overrightarrow{J} \cdot d\overrightarrow{S}, \tag{2}$$

$$\text{“Circulation” of } \overrightarrow{B} \text{ around } C = \mu_0 \times \text{ flux of current through } S.$$

Expression (2) can be used to deduce the magnetic field generated by a steady current I flowing along a straight wire.

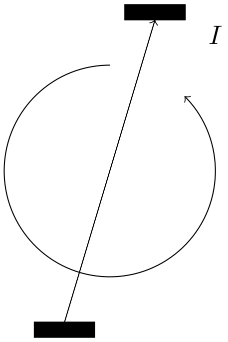

Figure 42:

A steady current I flows along a straight vertical wire (see Figure 43 above). The generated magnetic field acts only in the azimuthal direction (around the wire) and will have a constant magnitude at a fixed distance from the wire.

Assume that the wire is oriented in the z-direction and by symmetry the magnetic field $\overrightarrow{B}$ acts only in the azimuthal direction and depends only on the distance r from the wire, so

$$\overrightarrow{B} = (0, B_\phi(r), 0) \text{ in cylindrical polars}$$

Let C be a circle of radius R centred on the wire and in a constant z-plane. Now,

$$\begin{aligned} \oint_C \overrightarrow{B} \cdot d\vec{x} &= \oint_C (0, B_\phi((R), 0) \cdot (0, d\phi, 0) \\ &= B_\phi(R) \oint_C d\phi \\ &= B_\phi(R) 2\pi R, \end{aligned}$$

because the magnetic field is constant on the circle. But,

$$\mu_0 \int \int_S \overrightarrow{J} \cdot d\overrightarrow{S} = \mu_0 I,$$

and so by equation (2)

$$B_\phi(R) 2\pi R = \mu_0 I$$

$$\Rightarrow B_\phi(R) = \frac{\mu_0 I}{2\pi R},$$

the Biot-Savart law. The magnetic field is then

$$\overrightarrow{B} = \left(0, \frac{\mu_0 I}{2\pi R}, 0\right).$$

Summary

The Divergence Theorem of Gauss states that if V is the volume bounded by a closed surface S and $\vec{u}$ is a vector function of position with continuous derivatives, then

$$\int\int\int_V \nabla\cdot\vec{u}dV = \int\int_S \vec{u}\cdot\vec{n}dS$$

where $\vec{n}$ is the positive (outward drawn) normal to S.

Stokes' Theorem states that if S is an open, two-sided surface bounded by a closed, non-intersecting curve C (simple closed curve) then if $\vec{u}$ has continuous derivatives

$$\oint_C \vec{u}\cdot d\vec{r} = \int\int_S (\nabla\times\vec{u})\cdot\vec{n}dS$$

$$= \int\int_S (\nabla\times\vec{u})\cdot d\overrightarrow{S}$$

where C is traversed in the positive direction.

The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of the positive normal to S, has the surface on his left.

Green's theorem in the plane:

If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C Mdx + Ndy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

where C is traversed in the positive (counterclockwise) direction.

Green's theorem in the plane is a special case of Stokes' theorem. Also, it is of interest to notice that Gauss' divergence theorem is a generalization of Green's theorem in the plane where the (plane) region R and its closed boundary (curve) C are replaced by a (space) region V and its closed boundary (surface) S. For this reason the divergence theorem is often called Green's theorem in space.

Green's theorem in the plane also holds for regions bounded by a finite number of simple closed curves which do not intersect.

Post-Test

1. Verify Green's theorem in the plane for

$$\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy,$$

where C is the boundary of the region defined by:

   (a) $y = \sqrt{x}$, $y = x^2$;

   (b) $x = 0$, $y = 0$, $x + y = 1$

2. Evaluate

$$\int_{(1,0)}^{(-1,0)} \frac{-ydx + xdy}{x^2 + y^2}$$

along the following paths:

(a) straight line segments from $(1,0)$ to $(1,1)$, then to $(-1,1)$, then to $(-1,0)$.

(b) straight line segments from $(1,0)$ to $(1,-1)$, then to $(-1,-1)$, then to $(-1,0)$.

Show that although $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the line integral is dependent on the path joining $(1,0)$ to $(-1,0)$ and explain.

3. Verify the divergence theorem for $\vec{u} = 2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

4. Evaluate

$$\int\int_S \vec{r} \cdot \vec{n} dS$$

where

(a) S is the sphere of radius 2 with center at $(0,0,0)$,

(b) S is the surface of the cube bounded by $x = -1$, $y = -1$, $z = -1$, $x = 1$, $y = 1$, $z = 1$,

(c) S is the surface bounded by the paraboloid: $z = 4 - (x^2 + y^2)$ and the xy-plane.

5. Verify Stokes' theorem for $\vec{u} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$, where S is the surface of the cube $x = 0$, $= 0$, $z = 0$, $x = 2$, $y = 2$, $z = 2$ above the xy-plane.

6. Evaluate

$$\int\int_S (\nabla \times \vec{u}) \cdot \vec{n} dS,$$

where

$$\vec{u} = (x^2 + y - 4)\vec{i} + 3xy\vec{j} + (2xz + z^2)\vec{k}$$

and S is the surface of

(a) the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane

(b) the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

7. If $\vec{u} = 2yz\vec{i} - (x + 3y - 2)\vec{j} + (x^2 + z)\vec{k}$, evaluate

$$\int\int_S (\nabla \times \vec{u}) \cdot \vec{n} dS$$

over the surface of intersection of the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ which is included in the first octant.

8. A vector $\vec{u}$ is always normal to a given closed surface S. Show that

$$\int\int\int_V \text{Curl } \vec{u}\, dV = \vec{0},$$

where V is the region bounded by S.

Supplementary Reading

1. Hand, Louis N.; Finch, Janet D. Analytical Mechanics. Cambridge University Press.

2. Lanczos, Cornelius (1970). The variational Principles of Mechanics. Toronto: University of Toronto Press. ISBN 0-8020-1743-6.

3. Murray R.S. (1974). Schaum's Outline of Theory and Problems of Vector Analysis and an introduction to Tensor Analysis. McGraw-Hill, Inc.